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Systems & Information Theory - Noise, Bandwidth and Capacity

1.1 Introduction and Objectives:

Modern communication systems are required to transmit large amounts of data in the shortest possible time using the smallest possible bandwidth and be, as far as possible, immune to noise and interference. This section covers the basic principles of information transmission in the presence of noise and explores the limits of what is possible.

After studying this material, you will be able to:

Define the data rate of a signal transmitted by ideal rectangular symbols
Derive the Hartley-Shannon Law
Perform calculations to illustrate the trade-off between signal/noise ratio and bandwidth

1.2 Reading list

You will find this material in any elementary comms textbook, especially the following

Usher and Guy, "Information and Communication for Engineers"
Bateman, A, " Digital Communications"
Petersen, D, " Audio, Video and Data Communications"

1.3 The nature of signals

Since it can be shown that any continuous or ***analog*** signal can be digitised and reconstructed without loss of essential information (the Sampling Theorem), it is convenient to concentrate on the transmission of discrete or ***digital*** signals. In practice, most modern communication systems are digital eg digital Radio and TV broadcasting, mobile phones, computer networks. In fact, some of these principles can also be applied to data storage systems such as computer memory, CDs, disks and tapes.

1.4 The Data Rate of a Digital Signal

Any signal must change to convey information, and these changes are coded into ***symbols,*** (a symbol is the smallest indivisible part of a signal).

> The **signalling rate** is given by the **number of symbols per second**, where the symbols may be binary or multilevel.

Examples of symbols are:

Alphabetic- 26 letters of the alphabet

Numerical - digits 0-9

Binary - bits 0,1

Morse code - dots, dashes and spaces

Digital speech samples - 256 levels

Since **k bits** define **2^k levels**, an n-level symbol is equivalent to[1]

$$\mathbf{k = \log_2 n} \quad \textbf{binary symbols (bits).} \qquad \mathbf{1}$$

Hence a **256-level** digital speech sample is equivalent to **8 bits**. It is convenient for the purposes of comparison to convert symbols into the equivalent number of bits, and then

> The **data rate** is defined as the **number of bits per second** (or equivalent-bits per second)

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[1] Strictly this is the nearest integer *above* $\log_2 n$, since we cannot have a fraction of a symbol.

If the time allocated to a symbol is τ seconds, the signalling rate is $1/\tau$ symbol/s, and the data rate is therefore

$$\mathbf{C = 1/\tau \log_2 n \text{ bits/s}} \tag{2}$$

Note that 1 symbol/s = **1 baud**, a unit often employed in data communications (and often misused). As we shall see, 1 bit/s may be, but seldom is, equal to 1 baud..[2]

1.5 Bandwidth, capacity and noise - Shannon's 1st Theorem

The bandwidth of a channel will always be limited by practical or legislative considerations, as will the available power of the signal. In addition every channel will be contaminated by electrical noise. These three factors – bandwidth, power and noise - will determine the theoretical capacity of the channel. As mentioned above, we will concentrate on digital signals.

For simplicity we assume the symbols to be rectangular with width τ and amplitude A_i.[3] *They are not necessarily binary.*

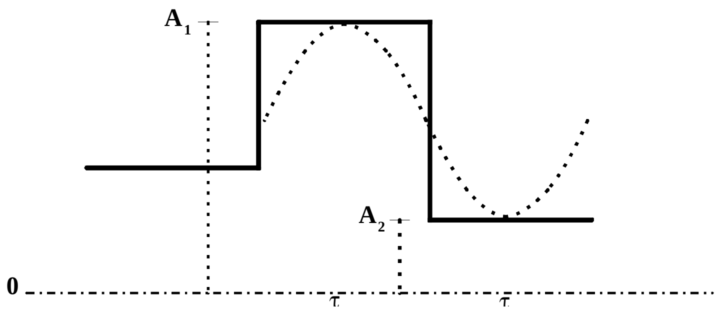

Figure 1.1

The Ideal Signalling Waveform

Fig 1.1 shows two successive symbols being transmitted at the fastest possible rate, with amplitudes (levels, values) $\mathbf{A_1}$ and $\mathbf{A_2}$.

Now, the *peak* values of one cycle of a sinusoid (offset by lower frequency components) can define the levels of the two successive rectangular symbols: if the maximum frequency that can be transmitted is B hertz (this is given by the bandwidth) then the maximum signalling rate ($^1/\tau$) is 2B baud.

For an acceptable error rate it can be shown that the *minimum separation between levels should be equal to the rms noise voltage* $(\sqrt{N})$. [4]

If maximum signal power is **S**, received signal power is **(S+N)** and peak amplitude for a rectangular waveform is $\sqrt{S+N}$.

Hence number of usable levels[5], each of which must be separated by $\sqrt{N}$, is

$$n = \frac{\sqrt{\mathbf{S+N}}}{\sqrt{\mathbf{N}}} \tag{3}$$

Hence the maximum data rate is

$$\boxed{\mathbf{C = B \log_2 (1 + S/N) \text{ bit/s}}} \tag{4}$$

This is the *Hartley-Shannon Law* for the capacity of a channel[6]. (Also known as *Shannon's Channel Capacity Theorem*).

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[2] The confusion arises because we use the same unit, the bit, for one binary symbol and also for for one unit of information. Hopefully the situation will be clarified after further study!

[3] A rectangular pulse is quite impractical, since it will require an infinite bandwidth. In practice, special pulse shapes are used which economise on bandwidth.

[4] It can be shown that this is a reasonable compromise between excessive power and unacceptable error rate.

[5] This is a greatly simplified treatment of the problem. A more rigorous derivation involves considerations of multi-dimensional signal space and hyper spheres. See the references for full details.

1.6 Limits to Performance - Shannon’s 2nd Theorem

The Hartley-Shannon Law (Shannon’s Channel Capacity Theorem) gives the theoretical capacity of a noisy communication channel, but has nothing to say as to how practical it is to achieve this performance.

As we have noted, random noise may produce errors in the reception of a signal, and if this is unacceptable then coding can be used for error detection and correction. As the theoretical capacity of the channel is approached, the error rate will increase and increasingly sophisticated coding methods will have to be employed. Shannon said that in theory we should be able to design a code which will permit transmission at the theoretical capacity with negligible error rate.

Thus *Shannon’s Channel Coding Theorem*:

> **“If the transmission rate is equal to or less than the channel capacity, then there exists a coding technique which enables transmission over the channel with an arbitrarily small frequency of errors”.**

The limiting channel capacity referred to in the Theorem is given by the Hartley-Shannon Law.

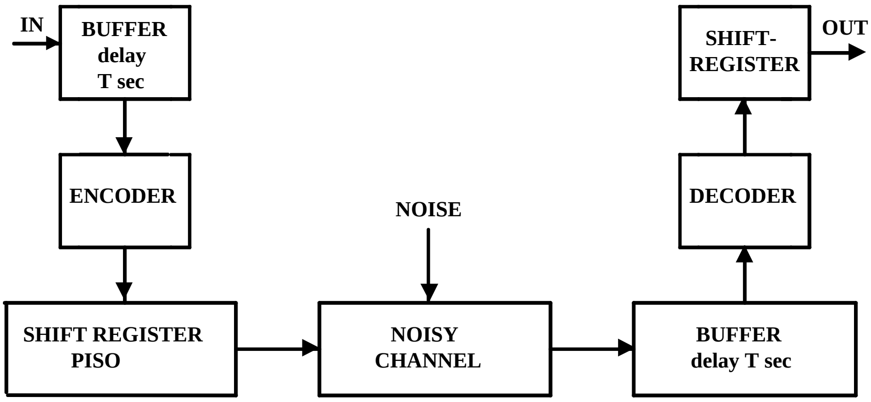

Figure 1.2 An Error Detection Coding Scheme

In general, coding will require storage and processing of the signals which causes the signals to be delayed. For very complex coding the buffer delays **T** become too long to be practical. Note that the coding technique is not specified by the theorem: it is up to engineers to design the codes:

1.7 The trade-off between Bandwidth and SNR

From the Hartley-Shannon Law

$$\mathbf{C = B \log_2 (1 + S/N)\ bit/s} \qquad 5$$

Hence **increased bandwidth can compensate for reduced SNR** and *vice-versa*. However, the relationship is obviously not linear.

We look at the model of a noisy, but otherwise ideal communication channel. The SNR will decrease along the channel to a minimum at the input to the receiver - $(S/N)_T$ - but the overall performance will be determined by the SNR at the receiver output, $(S/N)_R$. Because of coding and modulation, the transmitted bandwidth B_T will not usually be the same as the source bandwidth B_S. However, the receiver will restore the original bandwidth.

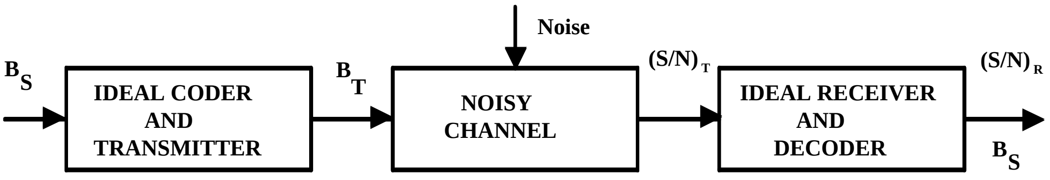

Figure 1.3 A Noisy Channel Model

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[6] Strictly this applies when the noise has a Gaussian Amplitude Distribution only, but this usually approximately true in practice.

For no loss of information, we have at the input and output of the receiver,

$$\mathbf{B_T \ \log_2 (1+(S/N)_T) = B_S \ \log_2 (1+(S/N)_R)}$$

and the overall SNR is therefore given by:

$$\mathbf{(S/N)_R \cong (S/N)_T^{\,B_T/B_S}} \qquad \mathbf{6}$$

Thus, there is an *exponential trade-off* between bandwidth and SNR, and it is more feasible to increase bandwidth, with a corresponding saving in SNR, than to reduce it. Note that in general, $B_T \geq B_S$ for practical systems.

1.8 A Realistic Noise Model

In the above analysis the noise power **N** was taken to be constant, but in real systems noise is usually assumed to be "white", and distributed over all frequencies with a continuous, flat spectrum .

The noise power within a bandwidth B is then given by

$$\mathbf{N_T = \eta B_T} \qquad \mathbf{7}$$

where η is the *Noise Power Spectral Density* or "V^2 /Hz" [7]

Hence

$$\mathbf{\left(\frac{S}{N}\right)_R \approx \left(\frac{S_T}{\eta B_S b}\right)^b} \qquad \mathbf{8}$$

where $\mathbf{b = \frac{B_T}{B_S}}$, the *Bandwidth Expansion Factor*.

This result is plotted in **Figure 1.4** (in dB) for comparison with the performance of practical systems, which often have a minimum threshold for acceptable performance.

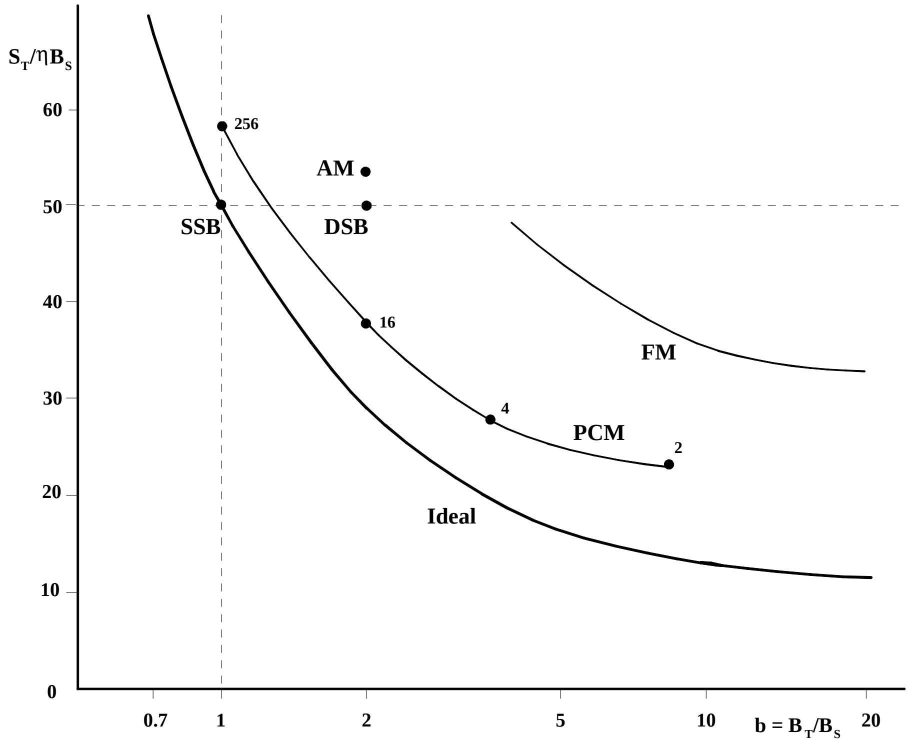

Fig. 1.4 Transmitted SNR Required for 50 dB Received SNR

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[7] Power, of course, is measured in watts, but in communication systems we use a normalised 1 Ω impedance so that 1 W = 1 V^2.

1.9 The Optimum Channel - the "Space Channel"

The most efficient channel, ie one having the minimum power, thus requires *infinite bandwidth*.

Minimum power and maximum bandwidth are the conditions under which satellites and deep space probes operate - minimum power because of obvious energy supply problems, and maximum bandwidth because operating frequencies can be very high, using UHF or microwave-bands, where there is little competition for spectrum usage. The space channel is a *power-limited channel,* and is contrasted with many terrestrial radio or cable channels which are *bandwidth limited channels.*

1.10 Sources of Noise

Noise in communication systems arises from both internal and external causes. External noise can be from local interference, atmospheric effects or from galactic sources. Internal, or circuit noise is due to random quantum mechanical effects in the conducting electrons, and is often dependent on the temperature of the component.

1.10.1 Thermal Noise (Johnson Noise)

Thermal motion of electrons in a conductor produces a random current, and therefore a noise power input to a system. The noise power is found to be proportional to the absolute temperature (T kelvin), and has a flat spectrum, at least up to 10^{12} Hz (1000 GHz) - hence the term *"White Noise"*.

Like all white noise, the total power is proportional to the bandwidth of the system, and is given by *Nyquist's Formula*:

$$\mathbf{N = kTB \ \ watts}^{8} \qquad \mathbf{9}$$

where **k** is *Boltzmann's Constant* and has the value 1.38 x 10^{-23}JK^{-1}.

In very sensitive applications such as a radio telescope, the input to the receiver is cooled with a liquid gas to reduce the thermal noise input

1.10.2 Antenna or "Sky" Noise

An antenna ("aerial") will receive noise from a variety of terrestrial, solar and galactic sources, depending on its angle of elevation, its bearing, time of day and sun-spot cycle, but primarily on the operating frequency. This sky noise can be represented by an increase in the thermal-noise temperature of the antenna.

Below **30 MHz** the noise is due to lightning discharges or "atmospherics": from **30 MHz** to **1 GHz** the noise is galactic ("cosmic") in origin. Between **1 GHz** and **10 GHz** the noise is generated within the atmosphere and a vertical antenna will receive less sky noise than a horizontal antenna. In this region the sky noise temperature can approach the minimum of 3K set by cosmic background radiation – the relic of the "Big Bang". The range from **2 – 8 GHz** is referred-to as the "low-noise" window, and is used for radio-telescopes and space telemetry. Above **10 GHz** the noise temperature rises in peaks due to resonance effects in water vapour and oxygen molecules, finally reaching a steady value around 290K.

An original reference:

Shannon C. E., "Communication in the Presence of Noise", Proc of the IRE, Vol. 37 pp10-21.

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[8] The noise power spectral density (PSD) is therefore η**= kT watts/herz**

Fig 1.5 Sky noise temperatures

Appendix 1 Frequency Bands and allocations

Frequency Band	*Wavelength*	*Noise source*	*Sky Noise Temp.*	*Application*
30 – 300 kHz	10 km – 1 km	Very high Atmospheric noise	> 10^8 K	LW broadcasting Radio beacons
300 – 3000 kHz	1000 m – 100 m	High Atmospheric noise	> 10^8 K	MW broadcasting Maritime radio
3 MHz – 30MHz	100 m – 10 m	Atmospheric noise	10^8 K - 10^5 K	SW Broadcasting Radio-telephone
30MHz – 300MHz	10 m – 1 m	High Galactic noise	10^5 K – 10^3 K	FM broadcasting Aircraft radio
0.3 –3 GHz	1 m – 10 cm	(Galactic noise) Cosmic background	10^3 K – 10 K	TV, Radar, Mobile phones Microwave links, Microwave ovens
3 GHz – 30 GHz	10 cm – 1 cm	Atmos. thermal noise, O_2 H_2O resonance	10 K – 10^2 K	Satellite broadcasting Radar