Couplings of asynchronously tuned coupled microwave resonators

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Abstract: The paper presents a comprehensive treatment of the couplings of asynchronously tuned coupled microwave resonators. The derived formulation allows one to extract coupling coefficients in terms of four characteristic frequencies that can be easily determined numerically or experimentally. To demonstrate the applications, numerical examples are given for asynchronously tuned microstrip coupled open-loop and dual-mode resonators.

1 Introduction

Asynchronously tuned RF/microwave narrow-band band-pass filters exhibit some attractive characteristics that may better meet some demanding requirements for rapid developments of mobile communications systems [1–5]. These requirements are imposed to conserve valuable frequency spectrum and enhance the performance of the systems. In an asynchronously tuned filter, each of resonators may resonate at different frequencies. Hence, to achieve an accurate or first-passed filter design it is essential to characterise couplings of coupled resonators whose self-resonant frequencies are different. This would seem more important for new circuit technologies such as superconducting thin film and micromachined filters for which the post-tunings after fabrications are not convenient. However, there is a lack of detailed information on open literatures about this subject.

In general two eigen frequencies in association with the coupling between a pair of coupled resonators can be observed despite whether the coupled resonators are synchronously or asynchronously tuned. If the coupled resonators are synchronously tuned, the coupling coefficient can be extracted from these two frequencies that are easily identifiable either in experiments or in full-wave EM simulations [6, 7]. However, if the coupled resonators are asynchronously tuned, a wrong result will occur if one attempts to extract the coupling coefficient by using the same formula derived for the synchronously tuned resonators. Therefore, other appropriate formulas than those presented in [6, 7] should be sought. It is the main purpose of this paper to present a comprehensive treatment of the couplings of asynchronously tuned coupled microwave resonators. The derived formulation allows one to extract coupling coefficients in terms of four characteristic frequencies that can be easily determined numerically or experimentally. Numerical examples of asynchronously tuned microstrip coupled open-loop and dual-mode resonators are described for demonstrations.

2 General theory of couplings

In general, the coupling coefficient $k$ of coupled microwave resonators which can be different in structure and can have different self-resonant frequencies as referred to Fig. 1 may be defined on the basis of a ratio of coupled energy to stored energy, i.e.

$$k = \frac{\int \int \int \varepsilon E_1 \cdot E_2 dv}{\sqrt{\int \int \int \varepsilon |E_1|^2 dv \times \int \int \int \varepsilon |E_2|^2 dv}} + \frac{\int \int \int \mu H_1 \cdot H_2 dv}{\sqrt{\int \int \int \mu |H_1|^2 dv \times \int \int \int \mu |H_2|^2 dv}}$$

where all fields are determined at resonance, and the volume integrals are over entire effective regions with permittivity of $\varepsilon$ and permeability of $\mu$. The first term on the right-hand side represents the electric coupling while the second term the magnetic coupling. It should be remarked that the interaction of the coupled resonators is mathematically described by the dot operation of their space vector fields, which allows the coupling to have either positive or negative sign. A positive sign would imply that the coupling enhances the stored energy of uncoupled resonators, whereas a negative sign would indicate a reduction.

Fig. 1 General coupled microwave resonators

Resonators 1 and 2 can be different in structure and have different resonant frequencies.

On the other hand, it would be much easier to find some characteristic frequencies that are associated with the...
couplings. The coupling coefficient can then be determined if the relationships between the coupling coefficient and the characteristic frequencies are established. In what follows we derive the formulation of such relationships. Before processing further, it might be worth pointing out that although the following derivations are based on lumped-element circuit models, the outcomes are also valid for distributed element coupled structures on a narrow-band basis.

3 Formulation for coupling coefficients

3.1 Electric coupling

In the case that only the electric coupling is concerned, an equivalent lumped-element circuit as shown in Fig. 2 may be employed to represent the coupled resonators. The two resonators may resonate at different frequencies of \( \omega_1 = (L_1C_1)^{-1/2} \) and \( \omega_2 = (L_2C_2)^{-1/2} \), respectively, and are coupled electrically through mutual capacitance \( C_m \). For natural resonance of the circuit of Fig. 2, the condition is

\[
Z_L = -Z_R
\]

where \( Z_L \) and \( Z_R \) are the input impedances when we look into the left and the right of reference plane \( T-T' \) of Fig. 2. The resonant condition of eqn. 2 leads to an eigen-equation

\[
\frac{1}{j\omega C_m} + \frac{j\omega L_1}{1 - \omega^2 L_1(C_1 - C_m)} + \frac{j\omega L_2}{1 - \omega^2 L_2(C_2 - C_m)} = 0
\]

After some manipulations the eigen-equation in eqn. 3 can be written as

\[
\omega^4 \left( L_1L_2C_1C_2 - L_1L_2C_m^2 \right) - \omega^2 \left( L_1C_1 + L_2C_2 \right) + 1 = 0
\]

We note that eqn. 4 is a biquadratic equation having four solutions or eigenvalues. Among those four we are only interested in the two positive real ones that represent the resonant frequencies which are measurable, namely,

\[
\omega_{1,2} = \sqrt{\left( L_1C_1 + L_2C_2 \right) \pm \sqrt{\left( L_1C_1 - L_2C_2 \right)^2 + 4L_1L_2C_m^2}} / 2\left( L_1L_2C_1C_2 - L_1L_2C_m^2 \right)
\]

The other two eigenvalues may be seen as their images. Define a parameter

\[
K_E = \frac{\omega_2^2 - \omega_1^2}{\omega_2^2 + \omega_1^2}
\]

where \( \omega_2 > \omega_1 \) is assumed. Since \( \omega_1 = (L_1C_1)^{-1/2} \) and \( \omega_2 = (L_2C_2)^{-1/2} \) we have by substitution

\[
K_E = \frac{C_m}{C_1C_2} \left( \frac{4}{\omega_1^2 + \omega_2^2} \right)^2 + \left( \frac{\omega_2^2 - \omega_1^2}{\omega_1^2 + \omega_2^2} \right)^2
\]

Now, define the electric coupling coefficient

\[
k_e = \frac{C_m}{\sqrt{C_1C_2}}
\]

\[
= \pm \frac{1}{2} \left( \frac{\omega_2^2 + \omega_1^2}{\omega_1^2 + \omega_2^2} \right)^2 \left( \omega_1^2 - \omega_2^2 \right) \left( \omega_2^2 + \omega_1^2 \right)
\]

in accordance of the ratio of the coupled electric energy to the average stored energy, where the positive sign should be chosen if a positive mutual capacitance \( C_m \) is defined.

3.2 Magnetic coupling

Shown in Fig. 3 is a lumped-element circuit model of asynchronously tuned resonators that are coupled magnetically, denoted by mutual inductance \( L_m \). The two resonant frequencies of uncoupled resonators are \( \omega_1 = (L_1C_1)^{-1/2} \) and \( \omega_2 = (L_2C_2)^{-1/2} \), respectively. The condition for natural resonance of the circuit of Fig. 3 is

\[
Y_L = -Y_R
\]

where \( Y_L \) and \( Y_R \) are the pair of admittances looked into the left and the right of reference plane \( T-T' \) of Fig. 3. This resonant condition leads to

\[
\frac{1}{j\omega L_m} + \frac{j\omega C_1}{1 - \omega^2 C_1(L_1 - L_m)} + \frac{j\omega C_2}{1 - \omega^2 C_2(L_2 - L_m)} = 0
\]

The eigen-equation in eqn. 10 can be expanded as

\[
\omega^4 \left( L_1L_2C_1C_2 - L_1L_2C_m^2 \right) - \omega^2 \left( L_1C_1 + L_2C_2 \right) + 1 = 0
\]

Similarly, this biquadratic equation has four eigenvalues, and the two positive real values of interest are

\[
\omega_{1,2} = \sqrt{\left( L_1C_1 + L_2C_2 \right) \pm \sqrt{\left( L_1C_1 - L_2C_2 \right)^2 + 4L_1L_2C_m^2}} / 2\left( L_1L_2C_1C_2 - L_1L_2C_m^2 \right)
\]

To extract the magnetic coupling coefficient we define a parameter

\[
k_m = \frac{\omega_2^2 - \omega_1^2}{\omega_2^2 + \omega_1^2}
\]
Assume $q_2 > q_1$ in eqn. 13, and recall $q_1 = (L_1 C_1)^{1/2}$ and $q_2 = (L_2 C_2)^{1/2}$ so that

$$K_m^2 = \frac{L_m^2}{L_1 L_2} \frac{4}{\left(\frac{\omega_{02}}{\omega_{01}} + \frac{\omega_{01}}{\omega_{02}}\right)^2} + \left(\frac{\omega_{02}^2 - \omega_{01}^2}{\omega_{02}^2 + \omega_{01}^2}\right)^2$$  \hspace{1cm} (14)$$

Defining the magnetic coupling coefficient as the ratio of the coupled magnetic energy to the average stored energy, we have

$$k_m = \frac{L_m}{\sqrt{L_1 L_2}}$$

$$= \pm \frac{1}{2} \left(\frac{\omega_{02}}{\omega_{01}} + \frac{\omega_{01}}{\omega_{02}}\right) \left(\frac{\omega_{02}^2 - \omega_{01}^2}{\omega_{02}^2 + \omega_{01}^2}\right)$$

Similarly, the choice of a sign depends on the definition of the mutual inductance, which is normally allowed to be either positive or negative that corresponds to the same or opposite direction of the two loops currents.

![Fig. 4](async_circ.png)

**3.3 Mixed coupling**

In many coupled resonator structures both the electric and the magnetic couplings exist. In this case we may have a circuit model as depicted in Fig. 4. It can be shown that the electric coupling is represented by an admittance inverter with $J = \alpha C_m$ while the magnetic coupling is represented by an impedance inverter with $K = \alpha L_m$. Note that the currents denoted by $I_1$, $I_2$ and $I_3$ are the external currents flowing into the coupled resonator circuit. According to the circuit model of Fig. 4, by assuming all internal currents flow outward each node we can define a definite nodal admittance matrix with a reference at node $0$.

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$  \hspace{1cm} (16)$$

with

$$y_{11} = j\omega C_1 + \frac{1}{j\omega(L_1 - L_m)}$$

$$y_{12} = y_{21} = -\frac{1}{j\omega(L_1 - L_m)}$$

$$y_{13} = y_{31} = -j\omega C_m$$

$$y_{22} = \frac{1}{j\omega L_m} + \frac{1}{j\omega(L_1 - L_m)} + \frac{1}{j\omega(L_2 - L_m)}$$

$$y_{23} = y_{32} = -\frac{1}{j\omega L_m}$$

$$y_{33} = j\omega C_2 + \frac{1}{j\omega(L_2 - L_m)}$$

For natural resonance, it implies that

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ for } \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$  \hspace{1cm} (17)$$

This requires that the determinant of admittance matrix to be zero, i.e.

$$\begin{vmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{vmatrix} = 0$$  \hspace{1cm} (18)$$

After some manipulations, we can arrive at

$$\omega^4(L_1 C_1 L_2 C_2 - L_m^2 C_1 C_2 - L_1 L_2 C_m^2 + L_m^2 C^2)$$

$$-\omega^2(L_1 C_1 + L_2 C_2 - 2L_m C_m) + 1 = 0$$  \hspace{1cm} (19)$$

This biquadratic equation is the eigen-equation for an asynchronously tuned coupled resonator circuit with the mixed coupling. One can immediately see that letting either $L_m = 0$ or $C_m = 0$ in eqn. 19 reduces the equation to either eqn. 4 for the electric coupling or eqn. 11 for the magnetic coupling, which is what should be expected. There are four solutions of eqn. 19. However, only the two positive ones are of interest, and they may be expressed as

$$\omega_1 = \sqrt{-\frac{\mathcal{R}_B - \mathcal{R}_C}{\mathcal{R}_A}} \quad \omega_2 = \sqrt{-\frac{\mathcal{R}_B + \mathcal{R}_C}{\mathcal{R}_A}}$$  \hspace{1cm} (20)$$

with

$$\mathcal{R}_A = 2(L_1 C_1 L_2 C_2 - L_m^2 C_1 C_2 - L_1 L_2 C_m^2 + L_m^2 C^2)$$

$$\mathcal{R}_B = (L_1 C_1 + L_2 C_2 - 2L_m C_m)$$

$$\mathcal{R}_C = \sqrt{\mathcal{R}_B^2 - 2\mathcal{R}_A}$$

Define

$$K_X = \frac{\omega_2^2 - \omega_1^2}{\omega_2^2 + \omega_1^2}$$  \hspace{1cm} (21)$$

For narrow-band applications we can assume that $(L_1 C_1 + L_2 C_2) \gg L_m C_m$ and

$$\frac{(L_1 C_1 + L_2 C_2)^2}{\sqrt{L_1 C_1 L_2 C_2}} \approx 1$$

the latter actually represents a ratio of an arithmetic mean to a geometric mean of two resonant frequencies. Thus, we have

$$K_m^2 = \frac{4L_1 C_1 L_2 C_2}{(L_1 C_1 + L_2 C_2)^2} k_2^2 + \frac{(L_1 C_1 - L_2 C_2)^2}{(L_1 C_1 + L_2 C_2)^2}$$  \hspace{1cm} (22)$$

in which

$$k_2^2 = \left(\frac{C_m}{C_1 C_2} + \frac{L_m}{L_1 L_2} - \frac{2L_m C_m}{\sqrt{L_1 C_1 L_2 C_2}}\right)$$

$$= (k_e - k_m)^2$$  \hspace{1cm} (23)$$

Now, it is clearer that $k_e$ is nothing else but the mixed coupling coefficient defined as

$$k_e = k_x - k_m$$

$$= \pm \frac{1}{2} \left(\frac{\omega_{02}}{\omega_{01}} + \frac{\omega_{01}}{\omega_{02}}\right) \times \sqrt{\left(\frac{\omega_2}{\omega_1} - \frac{1}{\omega_2}\right)^2 - \left(\frac{\omega_{02}}{\omega_{01}} - \frac{1}{\omega_{02}}\right)^2}$$  \hspace{1cm} (24)$$
4 Numerical examples

So far we have derived the formulas for extracting the electric, magnetic and mixed coupling coefficients in terms of four characteristic frequencies that can very easily be determined by full-wave EM simulations. It should be interesting to notice that the formulas of eqns. 8, 15 and 24 are all the same. Therefore we may use the universal formulation

\[ k = \pm \frac{1}{2} \left( \frac{\omega_{02}}{\omega_{01}} + \frac{\omega_{01}}{\omega_{02}} \right) \times \sqrt{\left( \frac{\omega_2^2 - \omega_1^2}{\omega_2^2 + \omega_1^2} \right)^2 - \left( \frac{\omega_{02}^2 - \omega_{01}^2}{\omega_{02}^2 + \omega_{01}^2} \right)^2} \]  

(25)

to extract the coupling coefficient of any two asynchronously tuned coupled resonators regardless of whether the coupling is electric, magnetic or mixed. Needless to say, the formulation is applicable for synchronously tuned coupled resonators as well, and in that case it degenerates to

\[ k = \pm \frac{\omega_2^2 - \omega_1^2}{\omega_2^2 + \omega_1^2} \]  

(26)

To demonstrate the applications of the derived formulation, let us first consider the coupled microstrip open-loop resonators in Fig. 5, where the resonators are supposed to be on a substrate with a relative dielectric constant \( \varepsilon_r \) and a thickness \( h \). Each of the open-loop resonators is basically a folded half-wavelength resonator, and hence exhibits a maximum electric field around the open gap and a maximum magnetic field on the opposite side. The two resonators having opposite orientations and a separation denoted by \( s \) are coupled through the fringing fields. This is a typical mixed coupling example because both the electric and magnetic couplings occur. For our purpose, the sizes of the two resonators are kept the same except for the open gaps indicated by \( g_1 \) and \( g_2 \) to allow having different self-resonant frequencies \( f_{01} \) and \( f_{02} \), respectively. The frequency responses of the coupled resonators were simulated using a full-wave EM simulator [8].

Fig. 5  Coupled microstrip open-loop resonators
Substrate with a relative dielectric constant \( \varepsilon_r \) and a thickness \( h \). It is allowed that \( g_1 \) \( \neq g_2 \) for asynchronous tuning.

Fig. 6 shows two typical simulated frequency responses, where the full line is for a synchronously tuned case when \( f_{01} = f_{02} \), while the dotted line is for an asynchronously tuned case when \( f_{01} \neq f_{02} \). In each case the two resonant frequency peaks that correspond to the two characteristics frequencies of the coupled resonators, i.e. \( f_1 \) and \( f_2 \), are clearly identifiable. More numerical results are listed in Table 1, and the coupling coefficients extracted using eqn. 25 are plotted in Fig. 7 where the axis of frequency ratio represents a ratio of \( f_{02} \) to \( f_{01} \). It seems that the asynchronous tuning in the given range has a very small effect on the coupling. This implies an advantage of using the coupled resonator structure of Fig. 5 to design asynchronously tuned microstrip filters because the coupling is almost independent of the asynchronous tuning, which makes both the design and the tuning much easier. It should be mentioned that if one tries to use eqn. 26 to extract the coupling coefficients a different and wrong conclusion would be drawn.

![Fig. 6 Typical simulated frequency responses of the coupled microstrip open-loop resonators of Fig. 5](image)

| Table 1: Numerical results of the coupled microstrip open-loop resonators of Fig. 5 for \( g_1 \) fixed by 0.4mm |
|---|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( s \) (mm) | \( g_1 \) (mm) | \( f_{01} \) (MHz) | \( f_{02} \) (MHz) | \( f_1 \) (MHz) | \( f_2 \) (MHz) |
| 1 | 0.4 | 1684.7 | 1684.7 | 1613.2 | 1715.7 |
| 1 | 1.0 | 1664.7 | 1714.7 | 1631.2 | 1747.5 |
| 1 | 2.0 | 1664.7 | 1774.05 | 1642.0 | 1795.5 |
| 1 | 3.0 | 1664.7 | 1826.9 | 1647.5 | 1845.0 |
| 2 | 0.4 | 1664.7 | 1664.7 | 1643.4 | 1666.1 |
| 2 | 1.0 | 1664.7 | 1714.7 | 1666.6 | 1722.9 |
| 2 | 2.0 | 1664.7 | 1774.05 | 1660.4 | 1778.4 |
| 2 | 3.0 | 1664.7 | 1828.9 | 1661.7 | 1832.1 |
| 3 | 0.4 | 1664.7 | 1664.7 | 1654.5 | 1674.9 |
| 3 | 1.0 | 1664.7 | 1714.7 | 1663.7 | 1716.8 |
| 3 | 2.0 | 1664.7 | 1774.05 | 1663.7 | 1775.05 |
| 3 | 3.0 | 1664.7 | 1828.9 | 1664.0 | 1829.8 |

![Fig. 7 Extracted coupling coefficients of the coupled microstrip open-loop resonators of Fig. 5](image)

The second example considered is the asynchronously tuned microstrip dual-mode resonator shown in Fig. 8, where \( a \) and \( b \) are the two orthogonal dimensions of the microstrip patch. As it is well known that when \( a = b \), two degenerate resonant modes which have the same self-reso-
nant frequencies but orthogonal to each other exist. The coupling between the two degenerate modes is introduced by cutting a corner as Fig. 8 shows. The two coupled modes function as a doubly-tuned resonator circuit. However, little is known about what would happen on the coupling when \( a \neq b \), in which case the two modes are asynchronously tuned. We investigated this issue with the aid of the full-wave EM simulator [8]. Table 2 lists some typical simulated characteristic frequencies of the resonator structure of Fig. 8 for \( b = 12 \text{mm} \) and \( d = 2 \text{mm} \). The simulations assumed a substrate with a relative dielectric constant \( \varepsilon_r = 10.8 \) and a thickness \( h = 0.635 \text{mm} \). Note that \( f_{01} \) is the self-resonant frequency of one orthogonal mode that is associated with the dimension of \( a \). While \( f_{02} \) is the self-resonant frequency of the other orthogonal mode associated with the dimension of \( b \). Therefore, \( f_{01} \) is shifted down when \( a \) is increased whereas \( f_{02} \) is unchanged for the fixed \( b \). For the comparison Fig. 9 shows the coupling coefficients calculated by eqns. 25 and 26, respectively. As can be seen the coupling tends to decrease when the two resonant modes are asynchronously tuned as predicted by eqn. 25. However, the wrong prediction with an opposite tendency was obtained by eqn. 26 because this formulation is only for the synchronously tuned coupled resonators.

![Fig. 8](image)

**Fig. 8** Microstrip dual-mode resonator

Substrate with a relative dielectric constant \( \varepsilon_r \) and a thickness \( h \). It is allowed that \( a \neq b \) for two asynchronously tuned coupled resonant modes.

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**Table 2:** Numerical results of coupled resonant modes of the microstrip dual-mode resonator for \( b = 12 \text{mm} \) and \( d = 2 \text{mm} \).

![Fig. 9](image)

**Fig. 9** Comparison of coupling coefficients extracted by eqns. 25 and 26

Asynchronously tuned resonant modes of the microstrip dual-mode resonator of Fig. 8

\( \square \) eqn. 25; \( \square \) eqn. 26

5 Conclusion

To achieve accurate or first-passed designs for asynchronously tuned RF/microwave narrow-band bandpass filters, it is essential to characterise couplings of coupled resonators whose self-resonant frequencies are different. For this purpose, this paper has presented a comprehensive treatment of the couplings of asynchronously tuned coupled microwave resonators. We have derived and arrived at a unique formulation that allows one to extract coupling coefficients of general coupled microwave resonators in terms of four characteristic frequencies, which can be easily determined numerically or experimentally. The detailed derivations would also provide a good tutorial on this subject. For the demonstration of applications, numerical examples of asynchronously tuned microstrip coupled open-loop and dual-mode resonators have been described.

6 References