ON THE DIFFERENT FORMS OF NAVIER-STOKES EQUATIONS

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Introduction

MATHEMATICAL PHYSICS

Famous Fluid Equations Are Incomplete

By NATALIE WOLCHOVER — 8 |

A 115-year effort to bridge the particle and fluid descriptions of nature has led mathematicians to an unexpected answer.

FLUID DYNAMICS

Mathematicians Find Wrinkle in Famed Fluid Equations

By KEVIN HARTNETT — 36 |

Two mathematicians prove that under certain extreme conditions, the Navier-Stokes equations output nonsense.

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Navier-Stokes Equations

■ **Continuity equation:**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho U] = 0, \qquad (1)$$

■ **Momentum balance equation:**

$$\frac{\partial \rho U}{\partial t} + \nabla \cdot [\rho U \otimes U] + \nabla \cdot [p \boldsymbol{I} + \boldsymbol{\Pi}^{(NS)}] = 0, \qquad (2)$$

■ **Energy balance equation:**

$$\frac{\partial}{\partial t}[\frac{1}{2}\rho U^2 + \rho \mathbf{e}_{in}] + \nabla \cdot [\frac{1}{2}\rho U^2 U + \rho \mathbf{e}_{in} U] + \nabla \cdot [(p\boldsymbol{I} + \boldsymbol{\Pi}^{(NS)}) \cdot U] + \nabla \cdot \boldsymbol{q}^{(NS)} = 0, \qquad (3)$$

$$\boldsymbol{\Pi}^{(NS)} = -2\mu \underbrace{\left[\frac{1}{2}(\nabla U + \widetilde{\nabla U}) - \frac{1}{3}\boldsymbol{I}\,(\nabla \cdot U)\right]}_{\overset{\circ}{\overline{\nabla U}}},$$

$$\boldsymbol{q}^{(NS)} = -\kappa \nabla T.$$

Thermo-mechanically consistent:

- **Galilean invariance**
- **Conservation of angular momentum**
- **Uniform center of mass motion, etc.,**

Recasting Methodology

■ The recasting methodology is based on a transformation technique which is similar in nature to that of Lorentz transformation.

■ It involves transforming the fluid velocity field variable within the standard fluid flow hydrodynamic equations.

■ We suggest the following forms of Navier-Stokes equations based on the mass, thermal and pressure diffusion effects.

- Mass-diffusion Navier-Stokes: $U \to U_v - \kappa_m \nabla \ln \rho$
- Thermal-diffusion Navier-Stokes: $U \to U_T - \kappa_T \nabla \ln T$
- Pressure-diffusion Navier-Stokes: $U \to U_p - \kappa_p \nabla \ln p$

■ κ_m, κ_T and κ_p are the molecular mass, thermal and pressure diffusivity co-efficients,respectively.

Recasted Navier-Stokes

■ Re-casted continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho U_v] = \kappa_m \Delta \rho, \qquad (4)$$

■ Re-casted momentum balance equation:

$$\frac{\partial \rho U_v}{\partial t} + \nabla \cdot [\rho U_v \otimes U_v] + \nabla \cdot \left[p\boldsymbol{I} + \boldsymbol{\Pi}_v^{(RNS)}\right] - \kappa_m^2 \nabla \Delta \rho + \kappa_m \nabla [\nabla \cdot (\rho U_v)] = 0, \qquad (5)$$

$$\boldsymbol{\Pi}_v^{(RNS)} = \boldsymbol{\Pi}_v + \frac{\kappa_m^2}{\rho} \nabla \rho \otimes \nabla \rho - \kappa_m U_v \otimes \nabla \rho - \kappa_m \nabla \rho \otimes U_v,$$

$$\boldsymbol{\Pi}_v = -2\mu \overset{\circ}{\overline{\nabla U_v}} + 2\mu\kappa_m \widetilde{\boldsymbol{D}} \ln \rho - \frac{2\mu}{3}\kappa_m \Delta \ln \rho \boldsymbol{I}.$$

Comparison with the Korteweg Tensor

■ Negative of the full pressure tensor:

$$\mathbf{T}^{(RNS)} = \left(-p - \frac{2}{3}\frac{\kappa_m \mu}{\rho^2}|\nabla \rho|^2 + \frac{2}{3}\frac{\kappa_m \mu}{\rho}\Delta \rho\right)\boldsymbol{I} + 2\frac{\kappa_m \mu}{\rho^2}\nabla \rho \otimes \nabla \rho + 2\mu \mathbf{D}(U_v) - \frac{2\mu}{3}(\nabla \cdot U_v)\boldsymbol{I} - \frac{\kappa_m^2}{\rho}\nabla \rho \otimes \nabla \rho - 2\frac{\kappa_m \mu}{\rho}\widetilde{\boldsymbol{D}}\rho + \kappa_m U_v \otimes \nabla \rho + \kappa_m \nabla \rho \otimes U_v \qquad (6)$$

■ The Korteweg tensor:

$$\mathbf{T} = (-p + \alpha_0 |\nabla \rho|^2 + \alpha_1 \Delta \rho)\boldsymbol{I} + \beta(\nabla \rho \otimes \nabla \rho) + 2\mu \mathbf{D}(U) + \lambda(\nabla \cdot U)\boldsymbol{I} \qquad (7)$$

■ Re-casted energy balance equation:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho U_v^2 + \rho \mathbf{e}_{in}\right] + \nabla \cdot \left[\frac{1}{2}\rho U_v^2 U_v + \rho \mathbf{e}_{in} U_v\right] + \nabla \cdot \left[(p\boldsymbol{I} + \boldsymbol{\Pi}_v) \cdot U_v - \kappa_m \boldsymbol{\Pi}_v \cdot \nabla \ln \rho\right] + \nabla \cdot \left[\boldsymbol{q}_v^{(RNS)}\right] + \nabla \cdot \left[\kappa_m \mathcal{N}_{v_1} + \kappa_m^2 \mathcal{N}_{v_2} + \kappa_m^3 \mathcal{N}_{v_3}\right] + \kappa_m \mathcal{N}_{v_4} + \kappa_m^2 \mathcal{N}_{v_5} + \kappa_m^3 \mathcal{N}_{v_6} = 0, \qquad (8)$$

$$\boldsymbol{q}_v^{(RNS)} = \boldsymbol{q}^{(NS)} - \kappa_m \rho \mathbf{e}_{in} \nabla \ln \rho - \kappa_m p \boldsymbol{I} \cdot \nabla \ln \rho, \qquad \mathcal{N}_{v_1} = -(U_v \cdot \nabla \rho) U_v - \frac{1}{2} U_v^2 \nabla \rho,$$

$$\mathcal{N}_{v_2} = (U_v \cdot \nabla \rho) \nabla \ln \rho + \frac{1}{2\rho}|\nabla \rho|^2 U_v, \qquad \mathcal{N}_{v_3} = -\frac{1}{2\rho}|\nabla \rho|^2 \nabla \ln \rho,$$

$$\mathcal{N}_{v_4} = \nabla \cdot \left[\rho U_v \otimes U_v + p\boldsymbol{I} + \boldsymbol{\Pi}_v^{(RNS)}\right] \cdot \nabla \ln \rho - U_v \cdot \left[\nabla \ln \rho \nabla \cdot (\rho U_v) - \nabla(\nabla \cdot (\rho U_v))\right],$$

$$\mathcal{N}_{v_5} = (U_v \cdot \Delta \rho \nabla \ln \rho) - (U_v \cdot \nabla \Delta \rho) + \frac{1}{2}\frac{|\nabla \rho|^2}{\rho^2}\nabla \cdot (\rho U_v), \qquad \mathcal{N}_{v_6} = -\frac{1}{2\rho^2}|\nabla \rho|^2 \Delta \rho.$$

Linear stability and sound dispersion of recasted Navier-Stokes

■ we consider re-casted Navier-Stokes models in a one dimensional flow configuration. A perturbation to the equilibrium ground state ρ_0, T_0, $p_0 = \boldsymbol{R}\rho_0 T_0$, $u_{v0} = 0$ is introduced as follows:

$$\rho = \rho_0(1 + \rho^*), \quad T = T_0(1 + T^*), \quad u_{v0} = u_{v0}^* \sqrt{\boldsymbol{R}T_0}, \quad p = p_0(1 + p^*) \text{ with } p^* = \rho^* + T^*. \qquad (9)$$

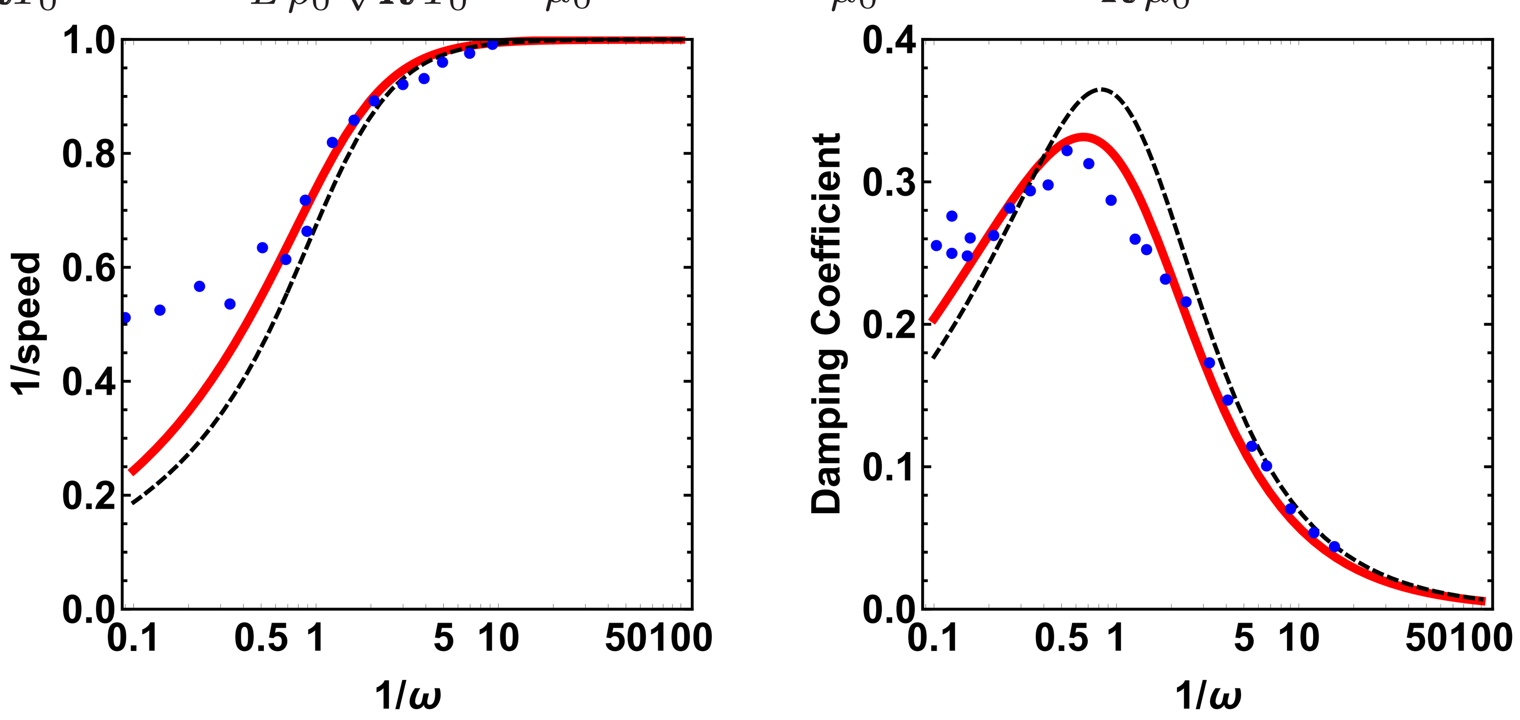
■ The dimensionless variables and the corresponding transport coefficients are:

$$x = Lx^*, \quad t = \tau t^*, \quad \tau = \frac{L}{\sqrt{\boldsymbol{R}T_0}}, \quad \mu^* = \frac{\mu}{L\rho_0\sqrt{\boldsymbol{R}T_0}} = \frac{\mu}{\mu_0}, \quad \kappa_m^* = \frac{\kappa_m \rho_0}{\mu_0}, \quad \kappa^* = \frac{\kappa}{\boldsymbol{R}\mu_0}. \qquad (10)$$

■ The dimensionless inverse of phase speed and dimensionless spatial damping coefficient are commonly defined as:

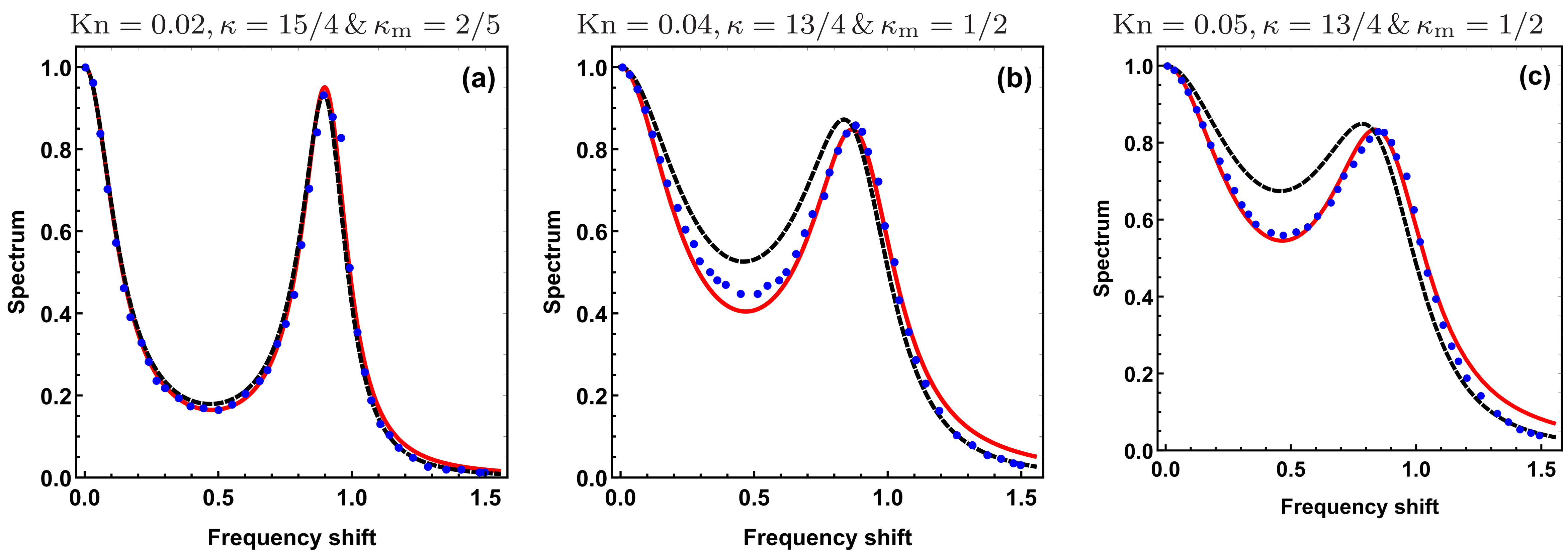
$$\sqrt{\frac{5}{3}}\frac{\mathrm{Re}[K]}{\omega} \quad \text{and} \quad -\sqrt{\frac{5}{3}}\frac{\mathrm{Im}[K]}{\omega}.$$

■ Figure: sound dispersion with $\kappa = 9/4$: inverse sound speed (left panel); damping coefficient (right panel). Solid line, dotted line and filled circles represents the results from the re-casted NS model, the classical NS model and the experimental results by Meyer & Sessler (1957), respectively.



Light Scattering Problem: spontaneous RBS spectra

■ The spectrum of the scattered light follows from the knowledge of the gas density fluctuations (the density-density correlation function) and are obtained from the linearized hydrodynamic models. The gas density fluctuations can either arise spontaneously or they can be created by external optical potentials.



Acknowledgements and References

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[1] S. K. Dadzie, J. Fluid Mech. 716, R6 (2013).
[2] H. Brenner, Physica A 349, 11 - 59 (2005).
[3] H. Brenner, Physica A 349, 60 - 132 (2005).
[4] E. Meyer & G. Sessler, Z. Phys. 149, 15 - 39 (1957).
[5] L. Wu et al., J. Fluid Mech. 763, 24 - 50 (2015).
[6] M. Heida & J. Malek, Int. J. Eng. Sci. 48, 1313 - 1324 (2010).