FINITE IMPULSE RESPONSE (FIR) DIGITAL FILTERS

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Chapter 6 of Textbook

(original material from John Thompson)

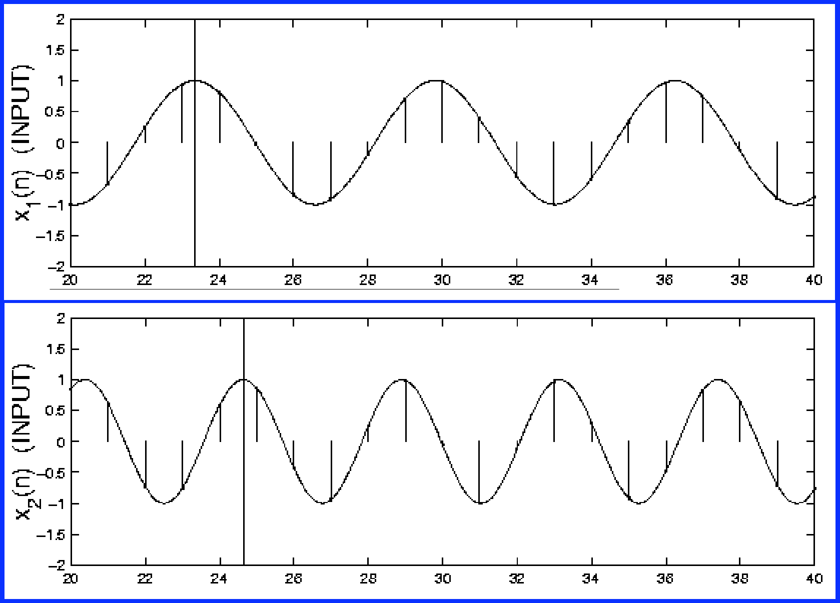
FIR Filters: Contents

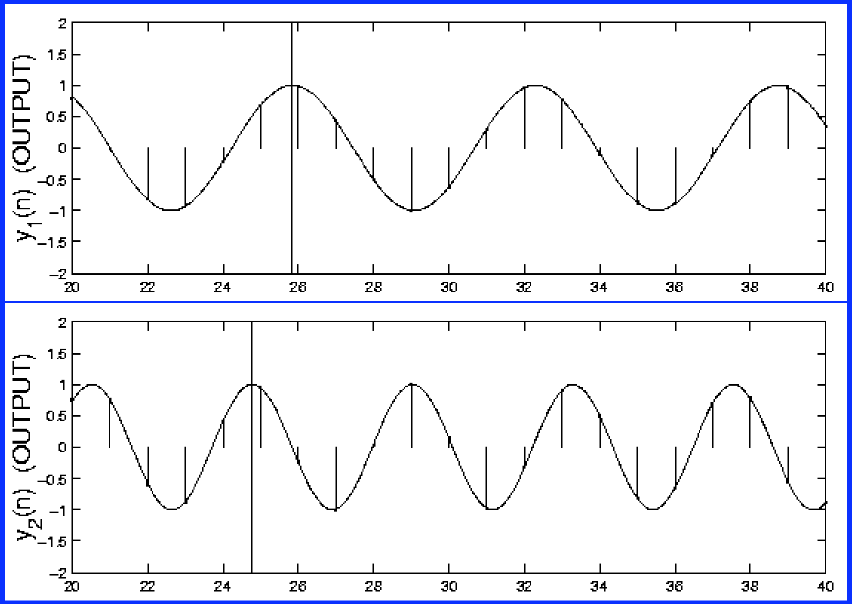
- Linear Phase FIR Filters
- Properties of FIR Filters
- Linear Phase FIR Design Example

IIR and FIR Filters

- IIR Filters:
  - Steep cut-off filter designs
  - Phase response is far from linear
- FIR Filters:
  - Phase response can be linear
  - Do not require high coefficient accuracy
  - Unconditionally stable
  - Less steep cut-off designs
  - More multiplies/adds required

Linear vs Nonlinear Phase Filtering





add the various frequency responses to obtain the input

nonlinear phase filtering misaligns the frequency responses of the output

Rectangular Pulse Responses

- Linear (Red) and Non-linear (Blue) phase:

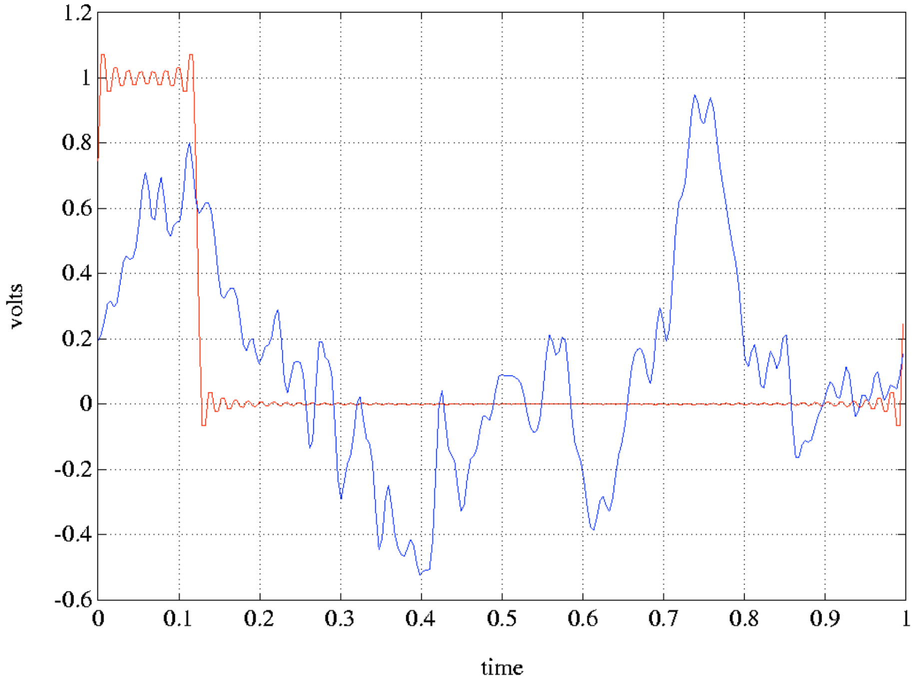


Fig 6.4b

When is Linear Phase Needed?

- Needed for reception with minimal distortion
  - Digital communications
  - Image processing
  - Audio signals

FIR Filter Operation

- Mathematical Equation
- Block Diagram

$$y(n) = \sum_{i=0}^{N-1} a_i x(n-i)$$

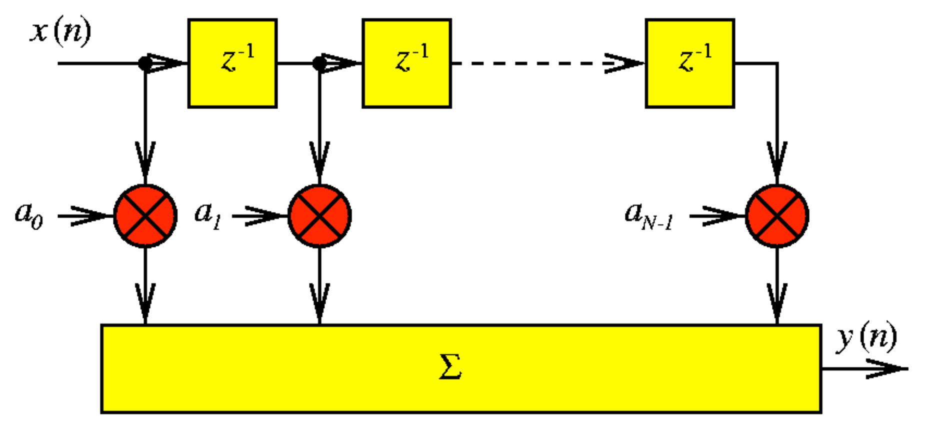


Fig 6.1

- *y(n)* is convolution of *{x(n)}* and *{a$_i$}*

FIR Transfer Function

- *z*-transform of equation:

$$Y(z) = \sum_{i=0}^{N-1} a_i X(z) z^{-i} = X(z) \sum_{i=0}^{N-1} a_i z^{-i}$$

- Hence the transfer function is:

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{i=0}^{N-1} a_i z^{-i}$$

Poles and Zeros

- Multiply *H(z)* by *(z^{N-1}/ z^{N-1})* yields:

$$H(z) = \frac{z^{N-1}}{z^{N-1}} \sum_{i=0}^{N-1} a_i z^{-i} = \frac{a_0 z^{N-1} + a_1 z^{N-2} + a_{N-1} z^{0}}{z^{N-1}}$$

- *H(z)* has *N-1* zeros and *N-1* poles
- All poles lie on the origin of z-plane
  - All FIR filters are unconditionally stable

FIR Filter Phase Response

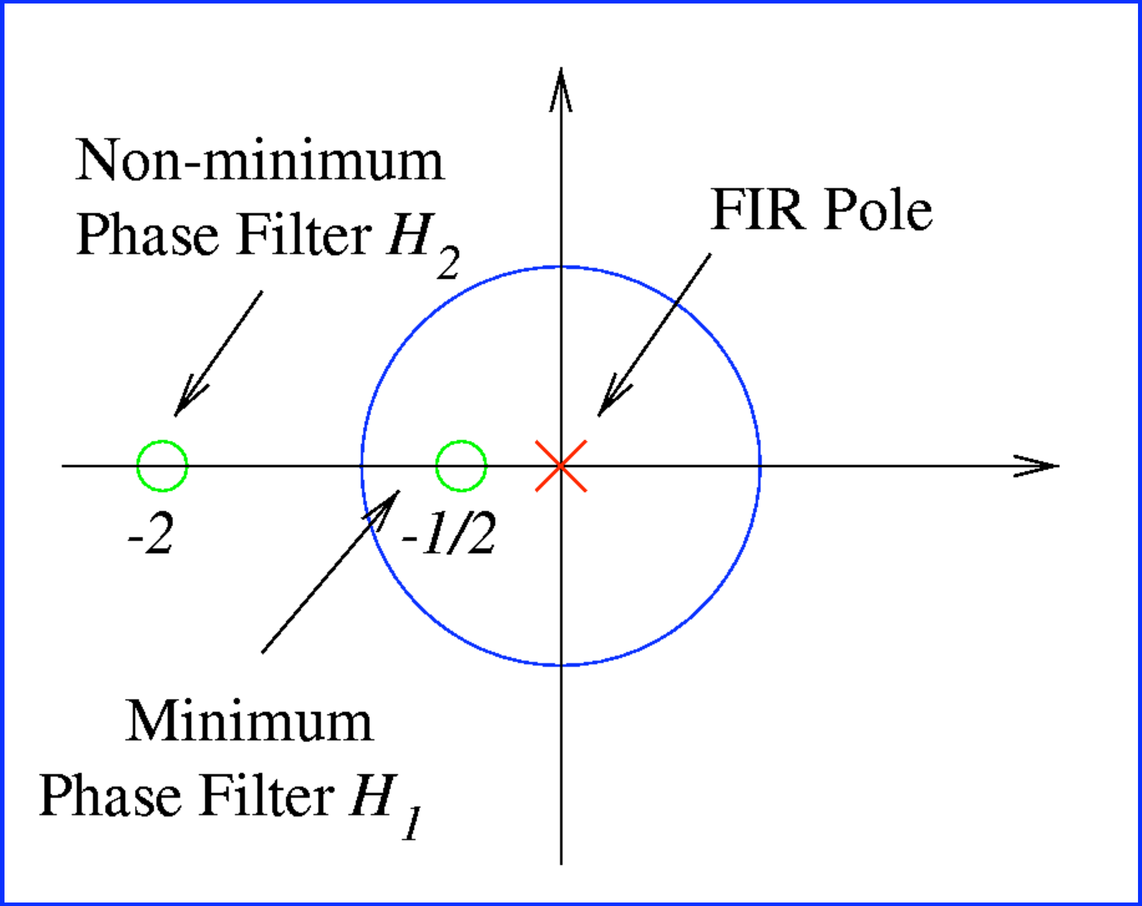
- Consider FIR Filter: $H_1(z) = 1 + z^{-1}/2$
  - Zero inside unit circle z = -1/2
  - Frequency response: $1 + \exp(-j\omega\Delta t)/2$

- Now consider: $H_2(z) = 1/2 + z^{-1}$
  - Zero now reflected in unit circle, z = -2
  - Frequency response: $1/2 + \exp(-j\omega\Delta t)$

Minimum Phase Filters

- Comparing H_1 and H_2
    - Magnitude of H_2 same as H_1 for all ω
    - But, phase shift of H_2 is bigger than H_1
    - H_2 has a bigger group delay
- Minimum phase/group delay solution:
    - Reflect zeros into unit circle

Z-Plane Plot of Filters



FIR Filter Frequency Response

- To obtain $H(\omega)$ replace z by $\exp(j\omega\Delta t)$:

$$\boxed{H(\omega) = \sum_{n=0}^{N-1} a_n \exp(-jn\omega\Delta t)}$$

- For any set of weights $\{a_i\}$, this equation gives $H(\omega)$

- This equation is a <u>Fourier series expansion</u> of $H(\omega)$

Fourier Series Coefficients

- $H(\omega)$ is periodic, with period $2\pi/\Delta t$

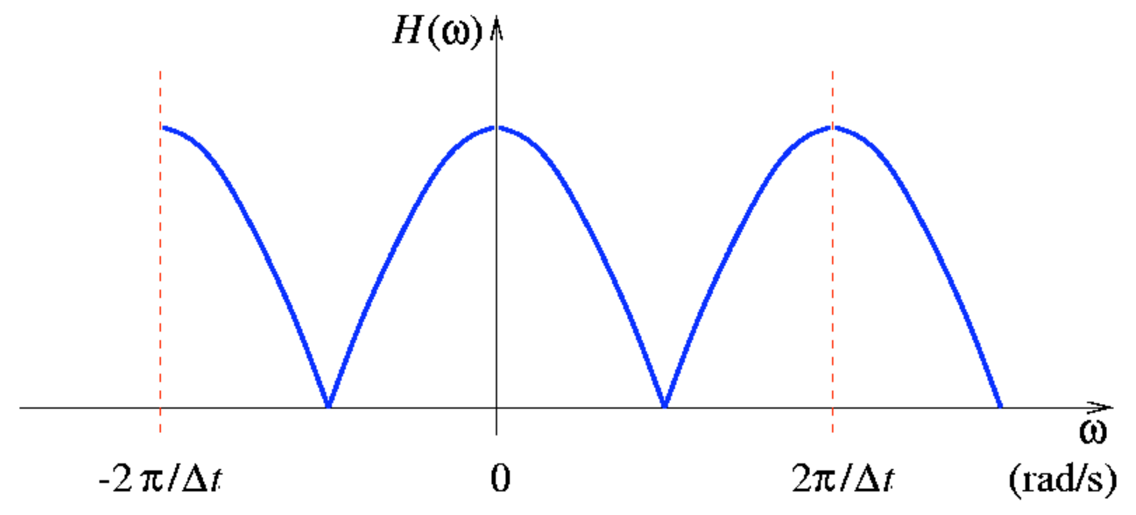


Fig 6.2

- Obtain

$$\boxed{a_n = \frac{\Delta t}{2\pi}\int_{0}^{2\pi/\Delta t} H(\omega)\exp(jn\omega\Delta t)\,d\omega}$$

Linear Phase FIR Filters

- If $H(\omega)$ is linear, can write:

$$H(\omega) = \underbrace{H_A(\omega)}_{\textit{Real amplitude}} \times \underbrace{\exp(-j\alpha\omega\Delta t)}_{\textit{Phase}}$$

- Key Properties of Filter:
  - Phase is linear function of ω
  - Delay of $\alpha\Delta t$ seconds at frequency ω

- Designing for linear phase:

$$H_A(\omega) = \sum_{n=-\infty}^{\infty} c_n \exp(-jn\omega\Delta t)$$

$$= \sum_{n=-\infty}^{\infty} c_n \cos(n\omega\Delta t) - jc_n \sin(n\omega\Delta t)$$

- Even symmetry removes imaginary part
  - So $c_n = c_{-n}$
  - Which yields: $H_A(\omega) = \sum_{n=-\infty}^{\infty} c_n \cos(n\omega\Delta t)$
- Now have a real amplitude function

Linear Phase Filters [Cont]

- $H_A(\omega)$ is an even function of ω

- In reality, use only 2M+1 taps:

$$H_A(z) = \sum_{n=-M}^{M} c_n z^{-n}$$

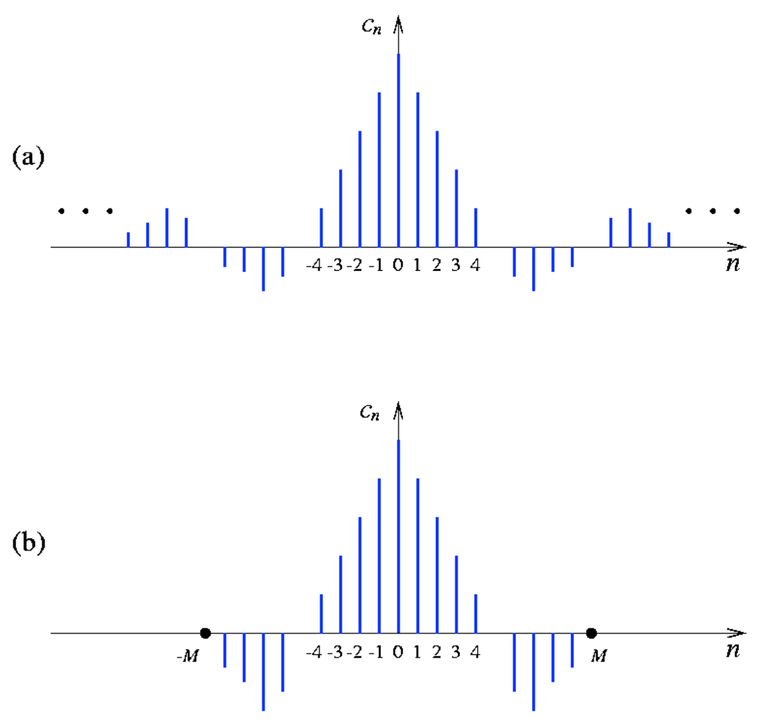


Fig 6.3

- Filter is non causal since the left side cannot be removed without removing the right side
- Non causal filters use samples from the future!

- Filter still not causal!

- Must use delay of α=*M* samples:

$$H(z) = z^{-M} H_A(z)$$

$$= \sum_{n=-M}^{M} c_n z^{-(n+M)}$$

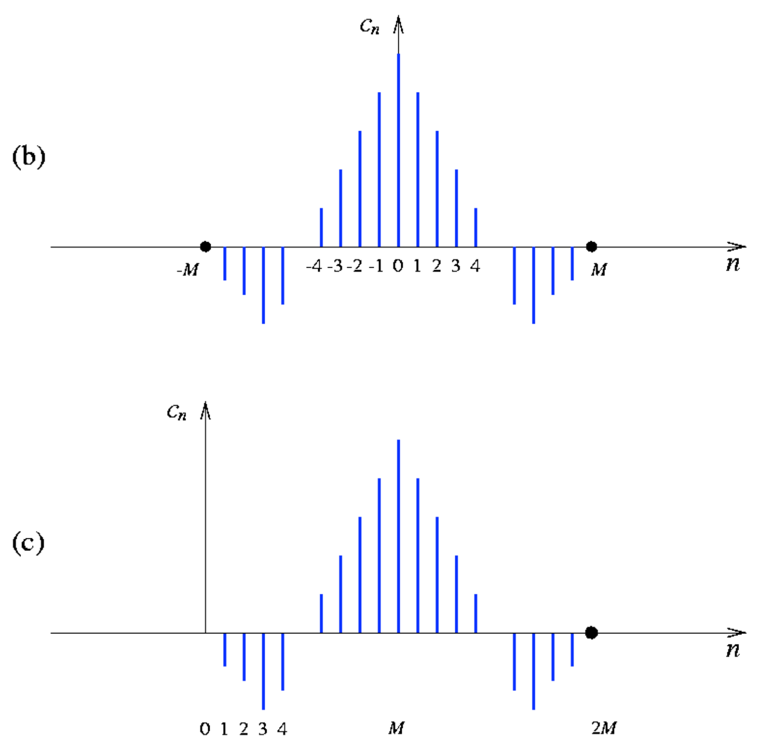


Fig 6.3

Linear FIR Summary

- In terms of original equation:

$$\underbrace{H(\omega) = \quad H_A(\omega)}_{\text{Amplitude}} \quad \times \quad \underbrace{\exp(-j\alpha\omega\Delta t)}_{\text{Phase}}$$

- Amplitude: Even symmetry $c_n = c_{-n}$
- Phase: Delay $\alpha = M$ samples

Linear Phase Filter Design

- Already seen that $H(\omega)$ is Fourier series of filter coefficients c_n
- So design an FIR filter as follows:
    - Specify our ideal filter $H(\omega)$
    - Calculate $2M+1$ Fourier coefficients $\{c_n\}$
    - Obtain Fourier Series approx of ideal $H(\omega)$

Obtaining Linear Phase Coefficients

- Use Fourier series equation:

$$c_n = \frac{\Delta t}{2\pi} \int_0^{2\pi/\Delta t} H(\omega) \exp(jn\omega\Delta t)\, d\omega$$

$$= \frac{\Delta t}{2\pi} \int_0^{2\pi/\Delta t} H(\omega) \cos(n\omega\Delta t)\, d\omega + j \frac{\Delta t}{2\pi} \int_0^{2\pi/\Delta t} H(\omega) \sin(n\omega\Delta t)\, d\omega$$

- c_n must be real so ignore sine term

- *H*(ω) is symmetric about π/Δ*t*, so:

$$c_n = \frac{\Delta t}{2\pi} \int_0^{2\pi/\Delta t} H(\omega)\cos(n\omega\Delta t)\,d\omega = \frac{\Delta t}{\pi} \int_0^{\pi/\Delta t} H(\omega)\cos(n\omega\Delta t)\,d\omega$$

- Coefficients c_n are symmetric, i.e $c_n=c_{-n}$
- So only need evaluate $n=0..M$ for $c_{-M}..c_M$
- Actual frequency response $H_A(\omega)$ is:

$$H_A(\omega) = \sum_{n=-M}^{M} c_n \cos(n\omega\Delta t) = c_0 + 2\sum_{n=1}^{M} c_n \cos(n\omega\Delta t)$$

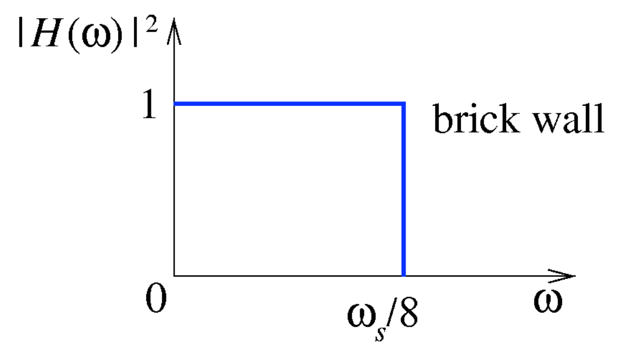
Design Example

Q: Design low pass FIR Filter for 1 kHz sample rate system, with gain 1 and a cutoff frequency of 125 Hz

Notice that:

$$\frac{\omega_s}{2\pi} = 1000$$

$$\frac{\omega_s}{2\pi \times 8} = 125$$



Use Fourier series equation to obtain filter

$$c_n = \frac{\Delta t}{\pi}\int_0^{\pi/\Delta t} H(\omega)\cos(n\omega\Delta t)\,d\omega$$

$$= \frac{\Delta t}{\pi}\int_0^{\omega_s/8} 1\times\cos(2\pi n\omega/\omega_s)\,d\omega$$

$$= \left[\frac{\Delta t}{\pi}\frac{\omega_s}{2\pi n}\sin\left(\frac{2\pi n\omega}{\omega_s}\right)\right]_0^{\omega_s/8}$$

$$= \frac{1}{\pi n}\sin\left(\frac{2\pi n\omega_s}{8\omega_s}\right) = \frac{1}{4}\times\frac{\sin(\pi n/4)}{\pi n/4} = \underline{\underline{\frac{\operatorname{sinc}(n/4)}{4}}}$$

Now for a 2M+1=21 tap filter:

$$c_0 = 0.250, c_1 = 0.225, \ldots, c_{10} = 0.0318$$

And to obtain filter coefficents use:

$$a_0 = a_{20} = c_{10} = 0.0318$$

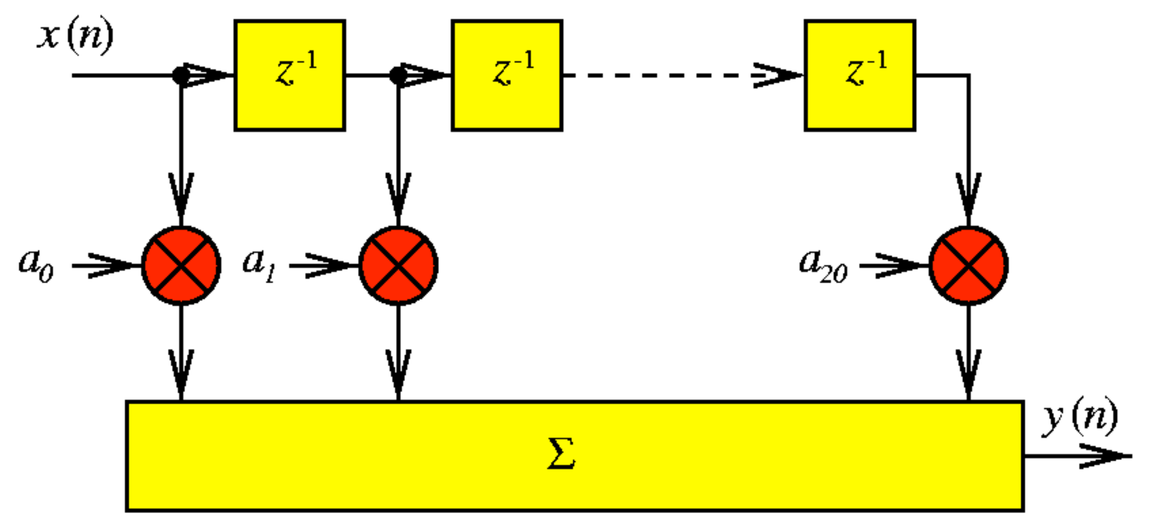
$$a_1 = a_{19} = c_9 = 0.025$$

$$\ldots\ldots\ldots$$

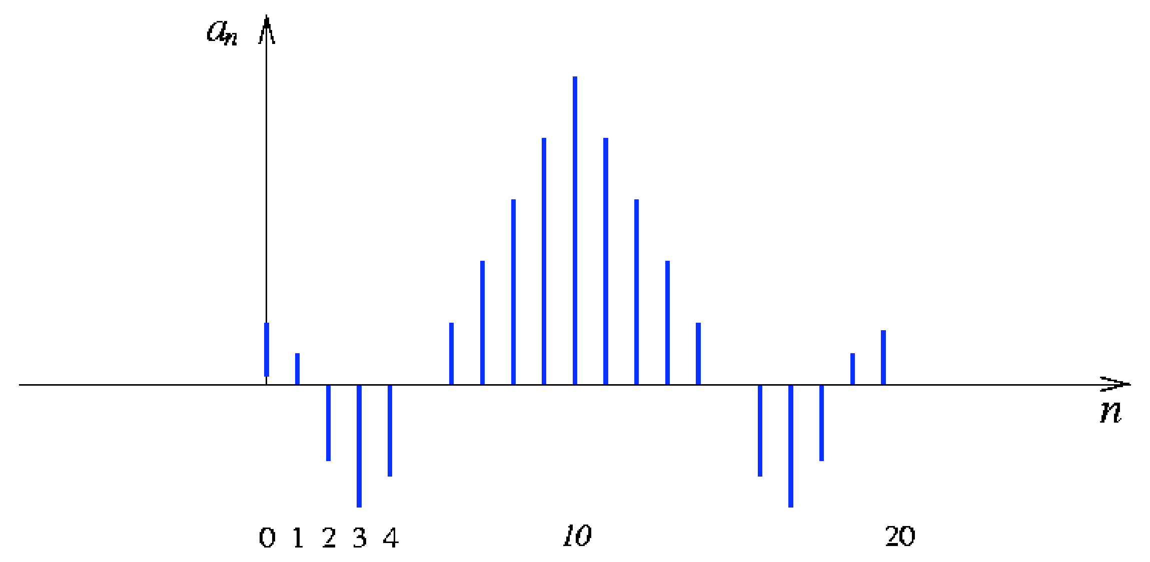
$$a_9 = a_{11} = c_1 = 0.225$$

$$a_{10} = c_0 = 0.250$$

Filter:



Coefficients:

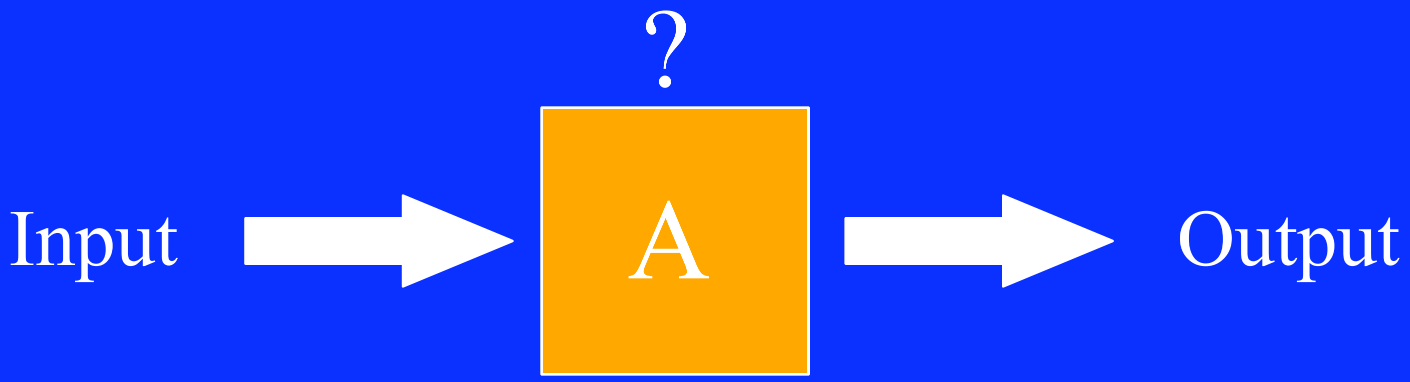


FIR Filters: Summary

- Linear vs Non-linear Phase Filters
- Frequency/Phase response of FIR filters
- Linear Phase FIR Design Example

Signal Processing -- Overview

- We have an unknown system A



Signal Processing -- Overview

- Use simple input functions (e.g. sines and cosines)
- Analyze simplest class of systems, i.e., linear systems
- Linear systems can be described by linear ordinary differential equations (ODEs)
- Linear ODEs can be solved by Fourier and Laplace analysis

Background Review

- Fourier Series
- Complex phasors
- Complex Fourier series

- Bonus: Dirac’s delta

Fourier Series

- Applies to finite-power periodic signals (period T) $w_0 = \frac{2\pi}{T}$

$$x(t) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(n w_0 t) + b_n \sin(n w_0 t)$$

$$\begin{aligned}
\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\cos(m w_0 t)dt \quad &= \quad \xcancel{\frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} a_0/2 \cos(m w_0 t)dt} \\
&+\frac{2}{T}\sum_{n=1}^{\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} a_n \cos(n w_0 t)\cos(m w_0 t)dt \\
&\xcancel{+\frac{2}{T}\sum_{n=1}^{\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} b_n \sin(n w_0 t)\cos(m w_0 t)dt} \\
&= \quad a_m
\end{aligned}$$

Complex Phasors

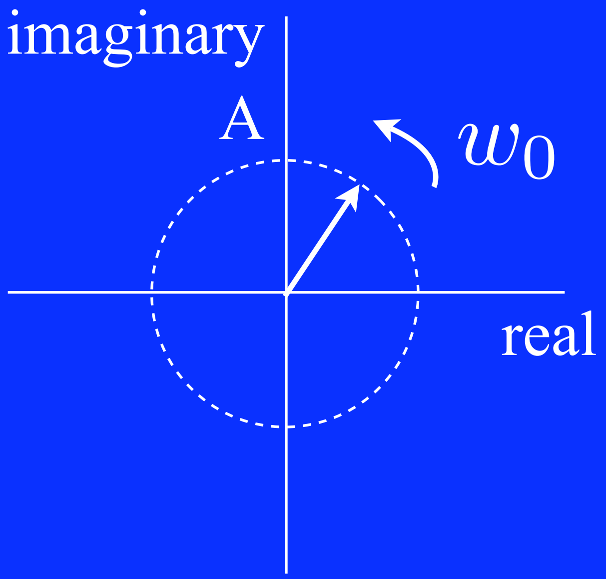
- Complex phasor: Ae^{jw_0t}
- Euler's identity: $e^{j\theta} = cos\theta + jsin\theta$
- Expansion of the complex phasor

$$Ae^{jw_0t} = A\cos(w_0t) + jA\sin(w_0t)$$

Complex Phasors

- Real and imaginary parts

$$\begin{aligned} Ae^{jw_0t} &= A\cos(w_0t) + jA\sin(w_0t) \\ &= \mathcal{R}(Ae^{jw_0t}) + j\mathcal{I}(Ae^{jw_0t}) \end{aligned}$$



Complex Fourier Series

- Applies to finite-power periodic functions

$$
\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} X_n e^{jnw_0 t} \\
&= \sum_{n=-\infty}^{\infty} X_n \left(\cos(nw_0 t) + j\sin(nw_0 t)\right)
\end{aligned}
$$

- Maps periodic functions onto discrete numbers

Dirac’s Delta

- It is the identity operator
- In the continuum:

$$x(t) = \int \delta(t-u)x(u)du$$

- In the discrete domain:

$$x(n) = \sum_i \delta(n-i)x(i)$$

$$\delta(n) = 1 \quad \text{iff } n = 0$$