FINITE IMPULSE RESPONSE (FIR) DIGITAL FILTERS

Paolo Favaro

Chapter 6 of Textbook -- part II

(original material from John Thompson)

FIR Filters: Contents

- Summarize Design Example
- Windowing in FIR Filters
- Examples and Use of Windows
- Minimax Filter Design

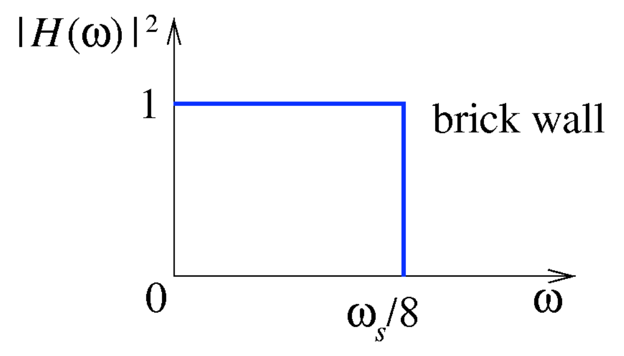
Design Example

Q: Design low pass FIR Filter for 1 kHz sample rate system, with gain 1 and a cutoff frequency of 125 Hz

Notice that:

$$\frac{\omega_s}{2\pi} = 1000$$

$$\frac{\omega_s}{2\pi \times 8} = 125$$



Use Fourier series equation to obtain filter

$$c_n = \frac{\Delta t}{\pi} \int_0^{\pi/\Delta t} H(\omega)\cos(n\omega\Delta t)\,d\omega$$

$$= \frac{\Delta t}{\pi} \int_0^{\omega_s/8} 1 \times \cos(n\omega\Delta t)\,d\omega$$

$$= \left[\frac{\Delta t}{\pi}\frac{\omega_s}{2\pi n}\sin\left(\frac{2\pi n\omega}{\omega_s}\right)\right]_0^{\omega_s/8}$$

$$= \frac{1}{\pi n}\sin\left(\frac{2\pi n\omega_s}{8\omega_s}\right) = \frac{1}{4} \times \frac{\sin(\pi n/4)}{\pi n/4} = 0.25\,\mathrm{sinc}(0.25n)$$

Now for a 2M+1=21 tap filter:

$$c_0 = 0.250, c_1 = 0.225, \ldots, c_{10} = 0.031$$

And to obtain filter coefficents use:

$$a_0 = a_{20} = c_{10} = 0.031$$

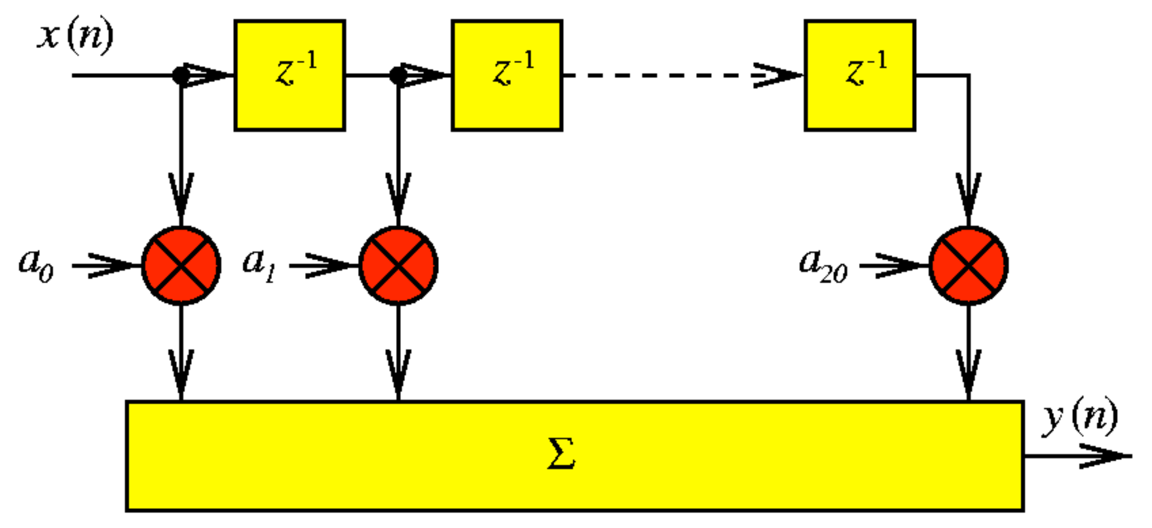
$$a_1 = a_{19} = c_9 = 0.025$$

$$\ldots\ldots\ldots$$

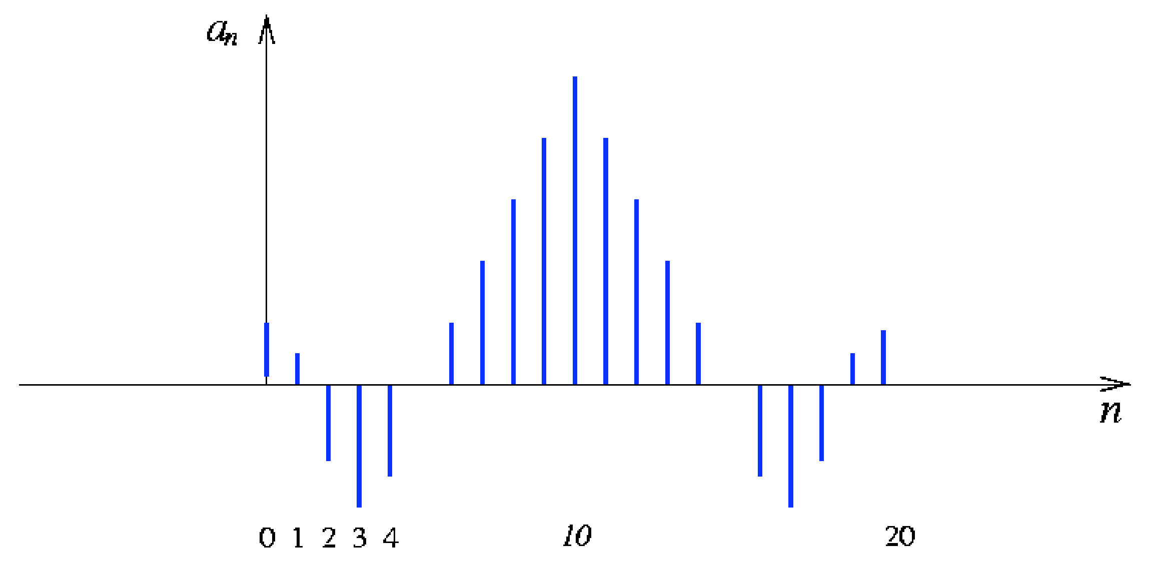
$$a_9 = a_{11} = c_1 = 0.225$$

$$a_{10} = c_0 = 0.250$$

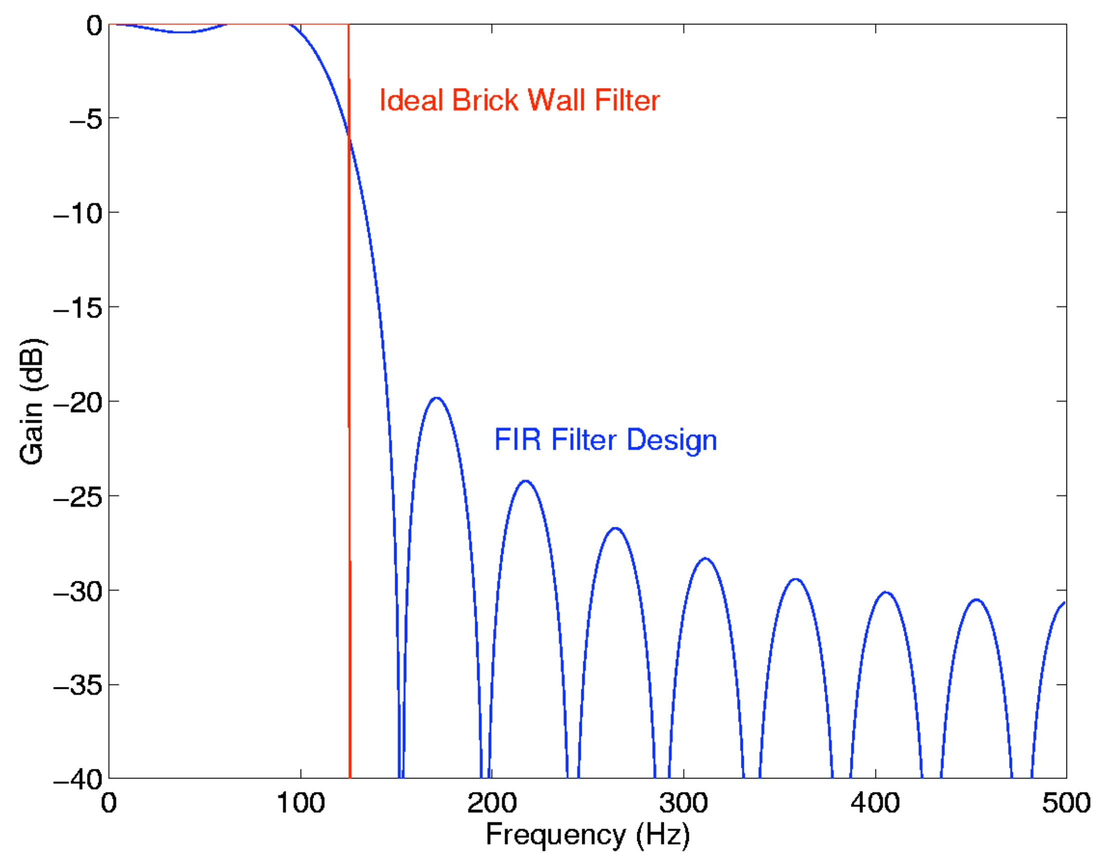
Filter:



Coefficients:



Filter Frequency Response



Windowing Effects

- We truncate infinite time sequence to sequence of length 2M+1 taps
- Equivalent to multiplying by rectangular window:

$$w_n = 1 \quad |n| \le M$$

$$w_n = 0 \quad |n| \ge M$$

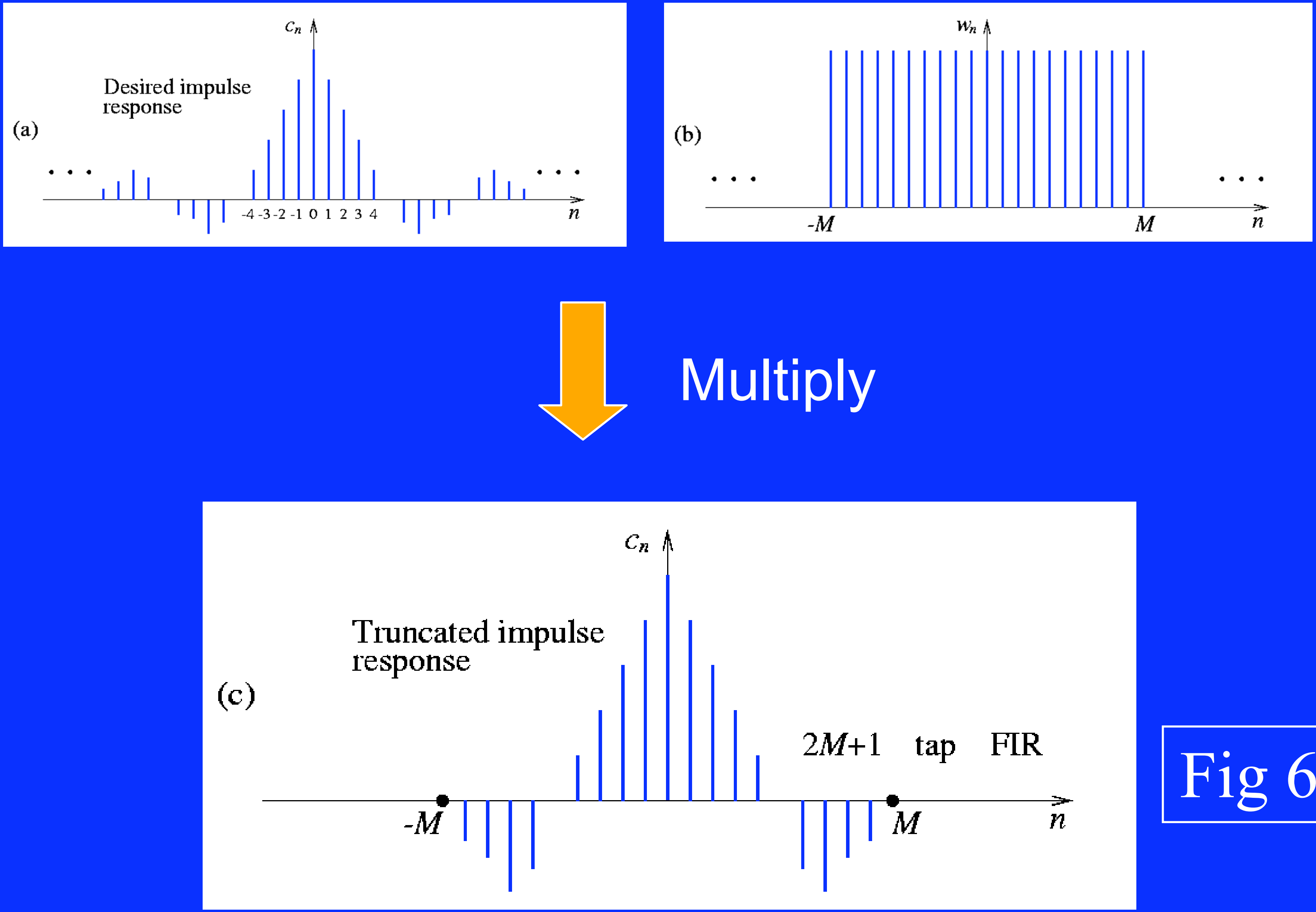
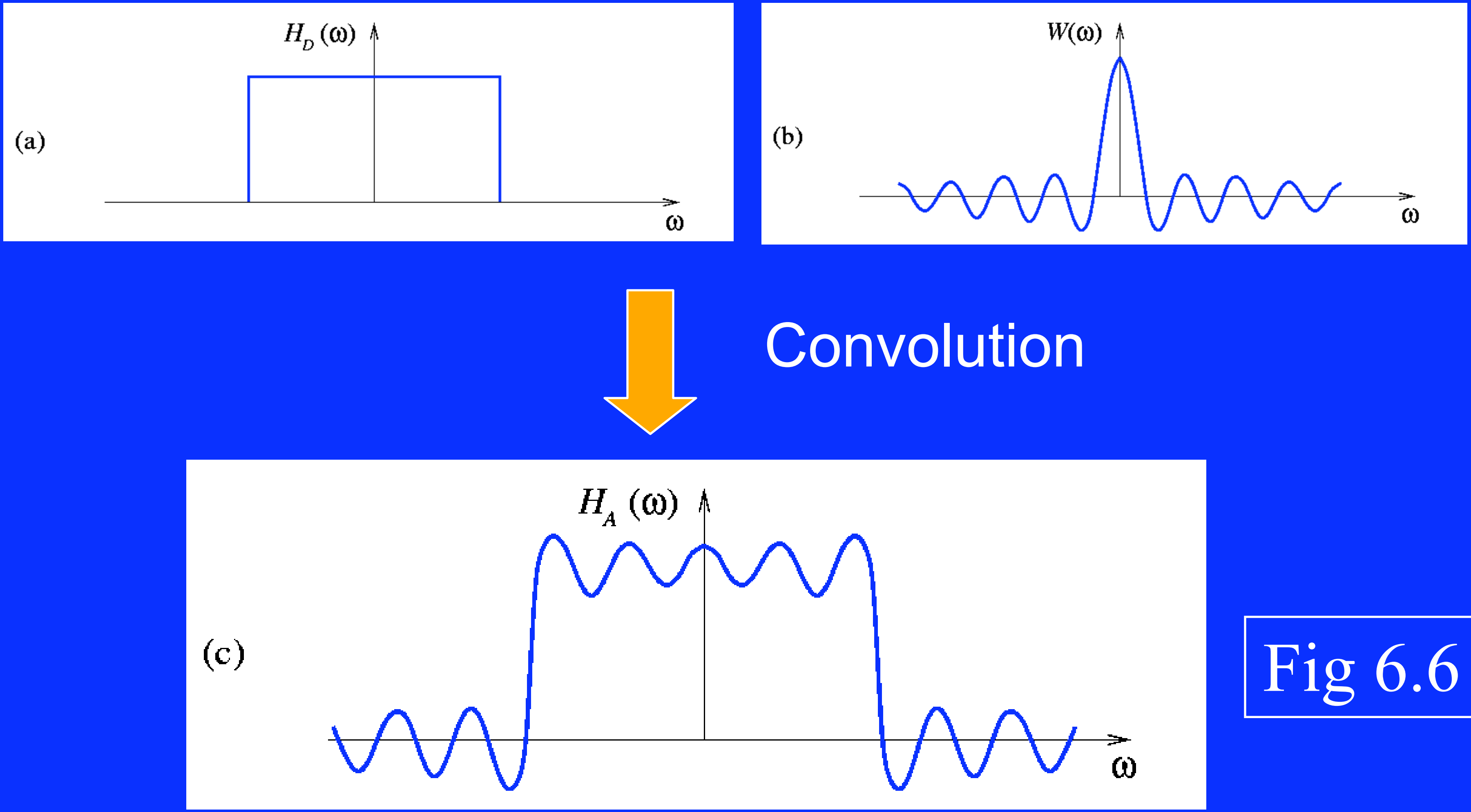

(a)
Desired impulse response
c_n
. . .
. . .
-4 -3 -2 -1 0 1 2 3 4
n
(b)
w_n
. . .
. . .
$-M$
M
n
Multiply
(c)
Truncated impulse response
c_n
$2M+1$ tap FIR
$-M$
M
n

Fig 6.5

Multiplication in time = Convolution in frequency:



Convolution

Fig 6.6

Windowing Summary

- Rectangular window has sinc function frequency response
- Sidelobes cause *ringing* of $H_A(\omega)$

- **Reminder:** $H_A(\omega)$ is given by

$$H_A(\omega) = \sum_{n=-M}^{M} c_n \cos(n\omega\Delta t)$$

Rectangular Window Example

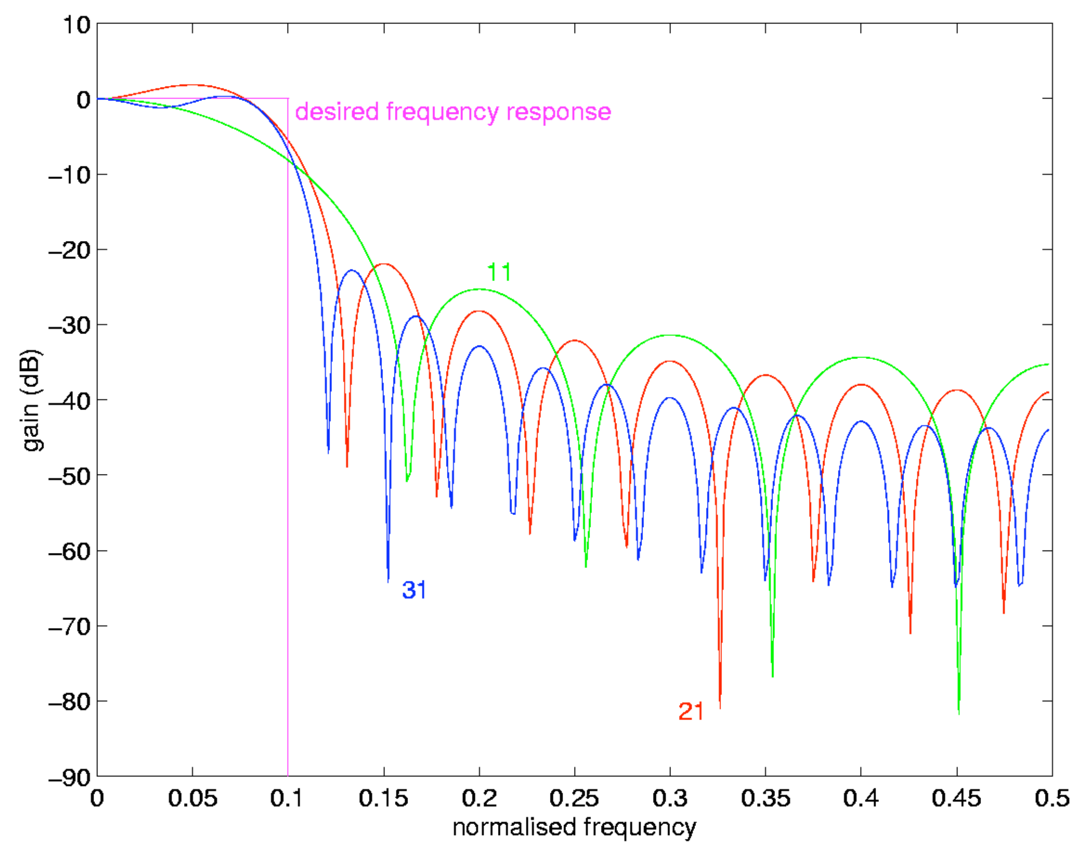


Fig 6.7

Alternative Window Functions

- Most important are $\cos^\alpha$ family
- Equation:

$$\boxed{w_n = \sin^\alpha(n\pi/N), \quad 0 \le n \le N-1}$$

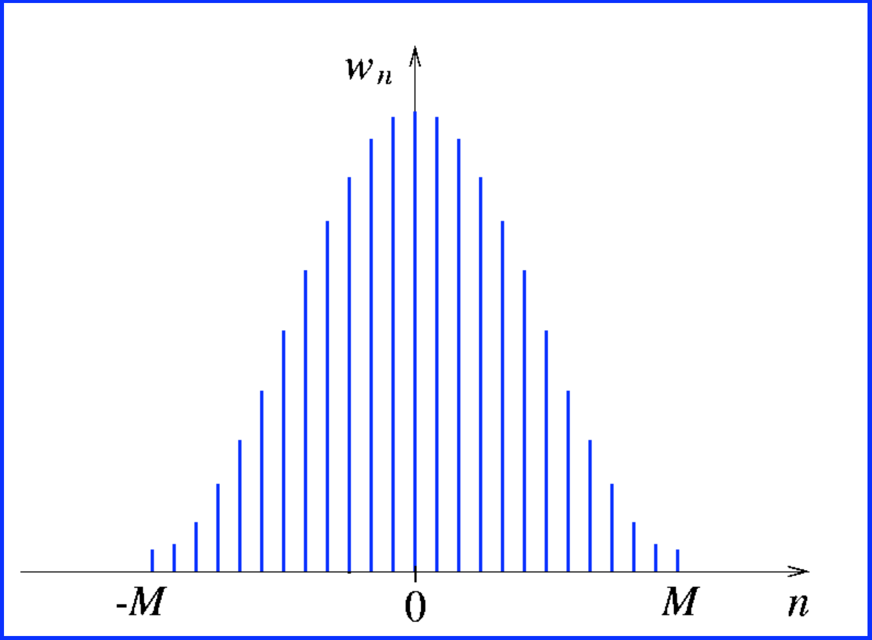
- Setting $\alpha=2$ yields *raised cosine window*:

$$\sin^2(n\pi/N) = 0.5[1.0 - \cos(2n\pi/N)]$$

- Convert limits |*n*|<*M*, with M=(N-1)/2
- Hanning Window: $w_n = 0.5 + 0.5\cos(n\pi / M)$
- Hamming Window: $w_n = 0.54 + 0.46\cos(n\pi / M)$

Hanning/Hamming Windows:

Fig 6.8



Comparison of Rectangular/Hamming windows:

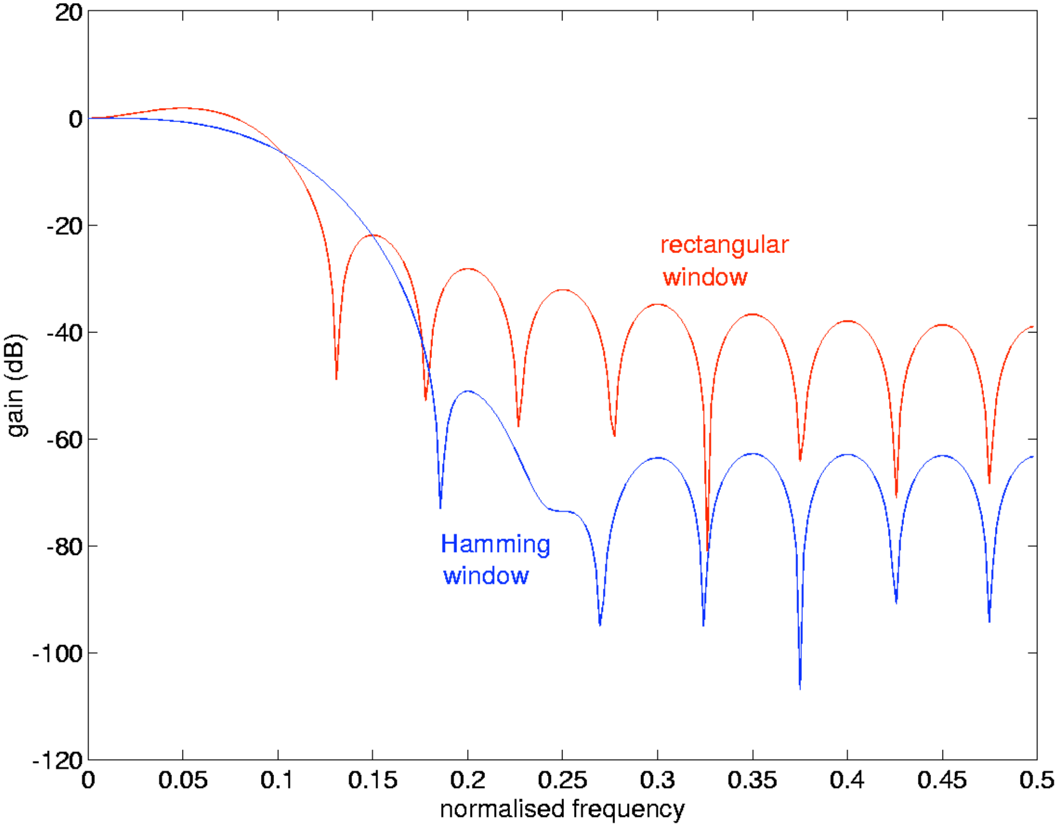
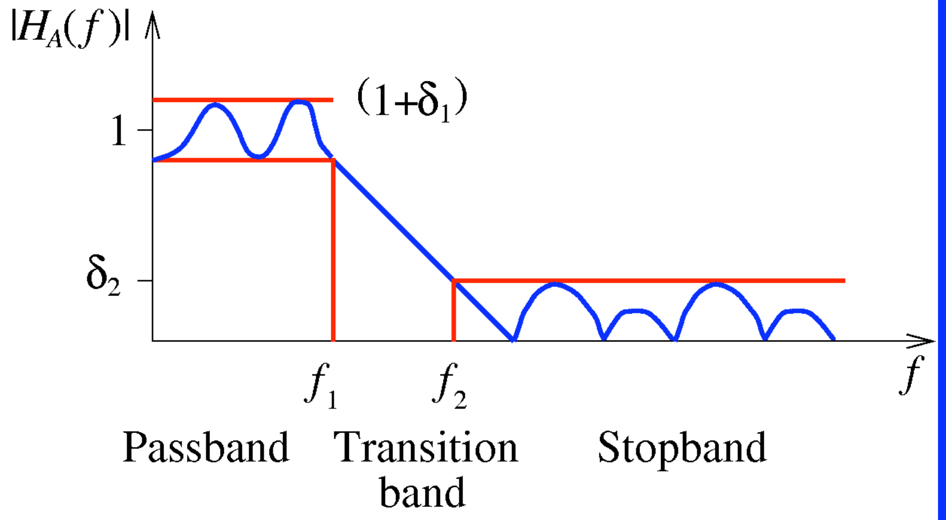


Fig 6.9

Window Characteristics



passband ripple = $20 \log_{10}(1+\delta_1)$ dB

stopband rejection = $20 \log_{10}(\delta_2)$ dB

transition band $\Delta f = f_2 - f_1$ Hz

Fig 6.10

Comparison of Windows (Table 6.1)

Window	**Transition Band (Hz)**	**Stopband Rejection**
Rectangular	$\frac{1}{N\Delta t}$	21 dB
Hanning	$\frac{3.1}{N\Delta t}$	44 dB
Hamming	$\frac{3.3}{N\Delta t}$	53 dB
Blackman	$\frac{5.5}{N\Delta t}$	74 dB

Selecting a Window Function

- Use stopband ripple to select window, eg 40 dB ripple → Hanning, Hamming
- Transition band Δ*f* specifies length *N*, e.g., for Hamming window:

$$\Delta f = \frac{3.3}{N\Delta t} \Rightarrow N = \frac{3.3}{\Delta f \Delta t}$$

FIR Filter Design Summary

- Determine the required stopband attenuation → specifies window function
- Determine No of taps *N=2M+1* from window's transition band function
- Calculate *M+1 (n=0...M)* coefficients:

$$c_n = \frac{\Delta t}{\pi} \int_0^{\pi/\Delta t} H(\omega)\cos(n\omega\Delta t)\,d\omega$$

- Apply window function w_n:

$$c'_n = c_n w_n$$

- Check actual frequency response:

$$H_A(\omega) = c'_0 + 2\sum_{n=1}^{M} c'_n \cos(n\omega\Delta t)$$

- Obtain 2M+1 tap values a_0-a_{2M} from coefficients c_n'

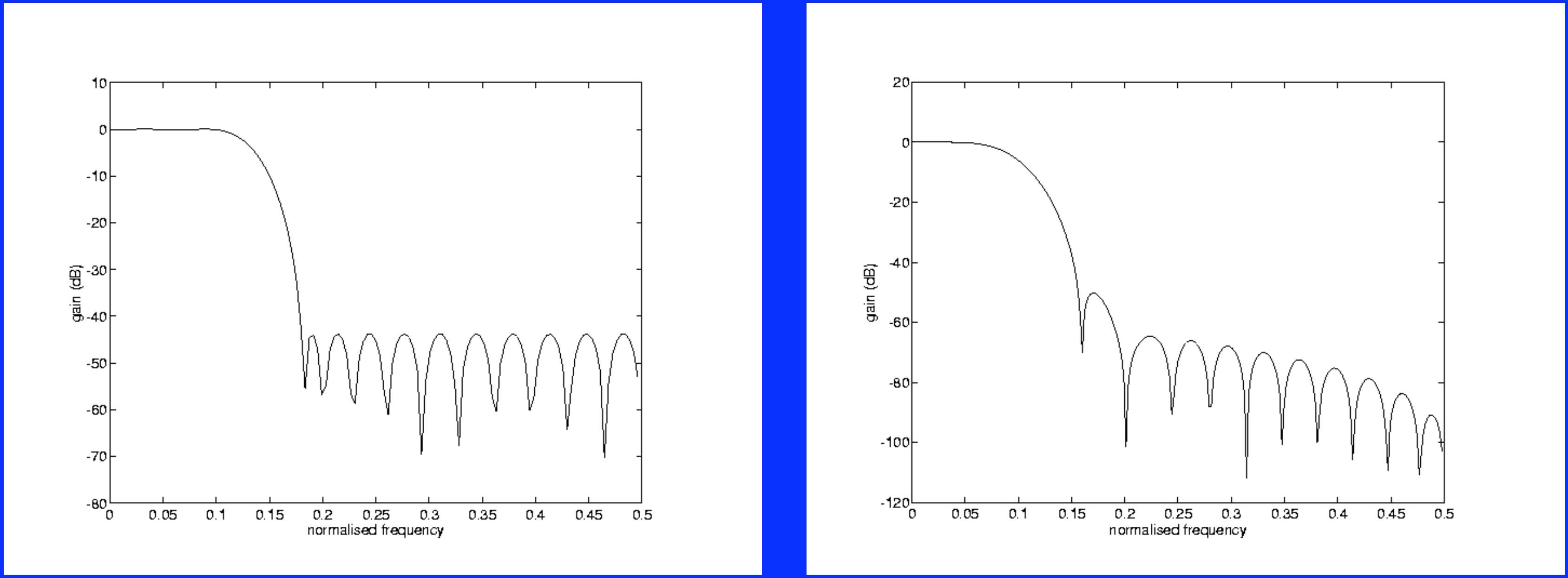
Minimax Filter Design

- Iteratively solve following equation:

$$\underset{\{c_n\}}{\text{minimum}} \left\{ \underset{\omega}{\text{maximum}} \, |L(\omega)[H_D(w) - H_A(w)]| \right\}$$

- MINImise MAXimum Error between desired and actual frequency response
- Reduces filter order required → widely used in commercial filter design

FIR Design Comparisons



20-tap Minimax

31-tap Hamming

Fig 6.11

Designing Minimax Filters

- Filter length for Minimax design:

$$N = \frac{2}{3}\frac{1}{\Delta t \Delta f}\log_{10}\left(\frac{1}{10\delta_1\delta_2}\right)$$

- Transition band Δf important
- Pass/Stop-band ripple δ_1, δ_2 have less effect

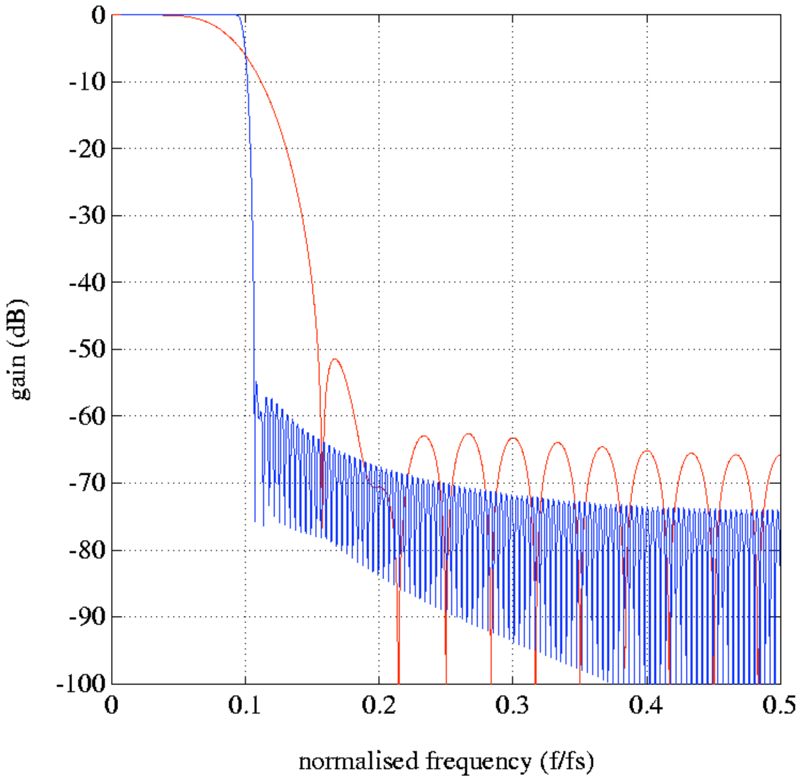


Fig 6.12

31-tap and 256-tap Minimax filter examples