THE DISCRETE FOURIER TRANSFORM (DFT)

Paolo Favaro

Chapter 9 of Textbook -- part I

(original material from John Thompson)

DFT: Contents

- Definitions of the DFT
- Properties of the DFT
- DFT Computation

The Continuous Fourier Transform

- The continuous Fourier transform of *x(t)* is defined as:

$$X(\omega) = \int_{0}^{\infty} x(t)\exp(-j\omega t)dt$$

- DSP Practice is different because:
  - *x(t)* is a sampled signal
  - Only have a finite No of samples of *x(t)*
- How does this affect estimation of *X*(ω)?

Fourier Transform of a Sampled Signal

- Assume we have an infinite number of samples of *x*, denoted as $x(n\Delta t)$
- The Fourier transform of *x* is given by:

$$X_c(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\exp(-j\omega n\Delta t)$$

- We still require an infinite No of samples of $x(n\Delta t)$ to use this equation!

Finite Data Record?

- Now only have N samples of $x(n\Delta t)$ from $n=0$ to $N-1$
- Can use this estimate of X_c:

$$\hat{X}_c(\omega) = \sum_{n=0}^{N-1} x(n\Delta t)\exp(-j\omega n\Delta t)$$

- How good is this estimate of X_c?

Windowing Functions

- The signal *x(t)* has been multiplied by a window function:

$$\hat{x}_c(t) = \sum_{n=0}^{N-1} x(n\Delta t)\delta(t - n\Delta t) = x_c(t) w_T(t)$$

- Multiplication in time is equivalent to convolution in the frequency domain:

$$\hat{X}_c(\omega) = \frac{1}{2\pi} X_c(\omega) * W_T(\omega)$$

Sampling/Windowing Effects

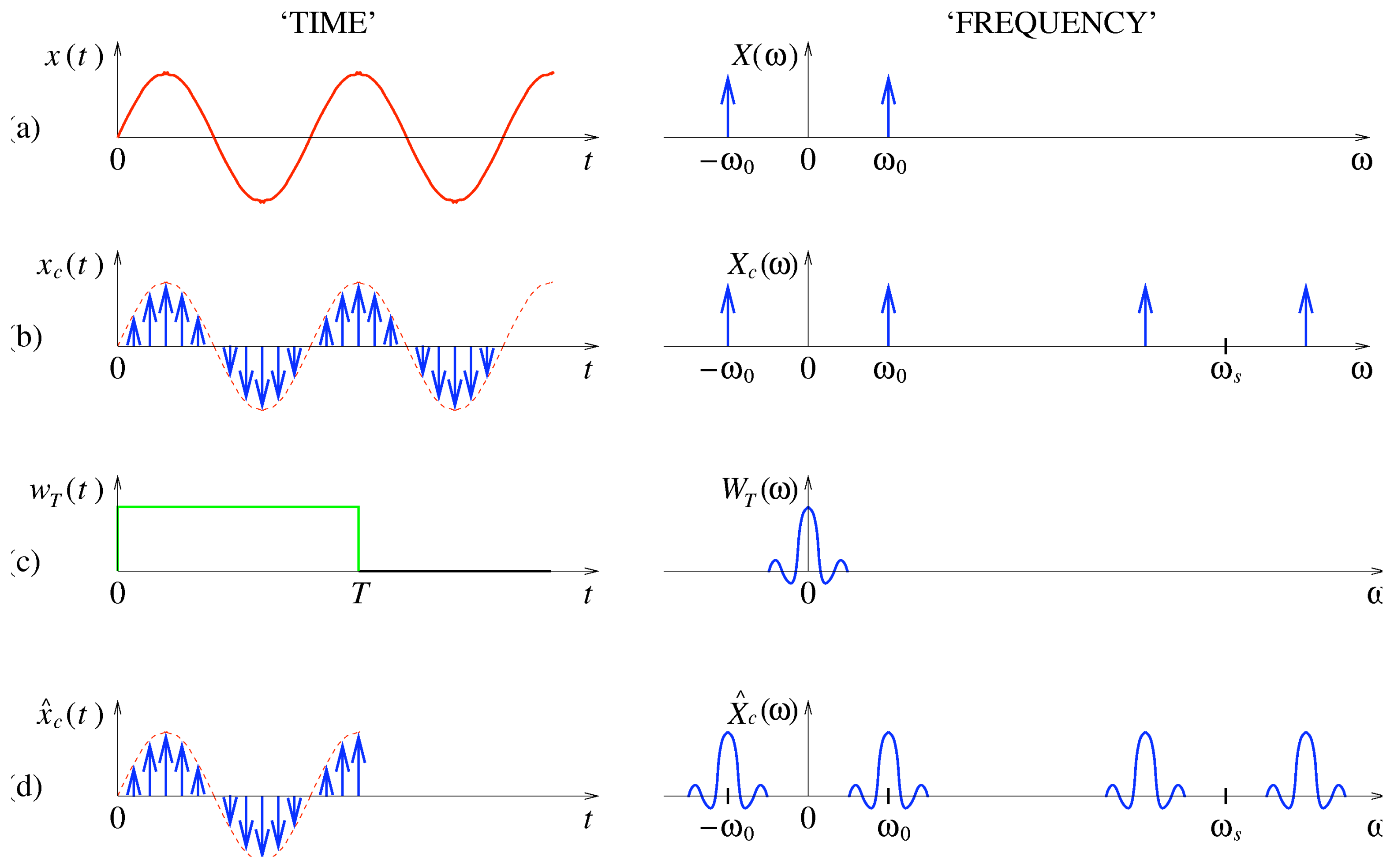


Fig
9.1

Discrete Fourier Transform

- $X_c(\omega)$ is a continuous function of the frequency ω
- It is usual to evaluate $X_c(\omega)$ at N equally spaced discrete frequencies, spaced by:

$$\Delta\omega = \frac{2\pi}{N\Delta t} \quad \text{or} \quad \Delta f = \frac{1}{N\Delta t}$$

- The frequency spacing can be reduced by increasing the number of samples N

Sampling $X_c(\omega)$ in Frequency

- The N samples of $X_c(\omega)$ can be written:

$$\hat{X}_c(k\Delta\omega) = \sum_{n=0}^{N-1} x(n\Delta t)\exp(-jk\Delta\omega n\Delta t),\; k = 0\ldots N-1$$

- Removing $\Delta\,\omega$ and Δt gives:

$$\hat{X}_c(k) = \sum_{n=0}^{N-1} x(n)\exp\left(\frac{-jnk2\pi}{N}\right)$$

- Typically $X_c(k)$ is written as $X(k)$

DFT Definition

- The DFT is defined as:

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\exp\left(\frac{-jnk2\pi}{N}\right) \quad 0 \le k, n \le N-1}$$

- The DFT inputs *N* samples *x(n)* and outputs *N* samples *X(k)*
- The integer *k* is the ‘bin’ number of the DFT output for frequency $k\Delta\omega$

DFT Bins

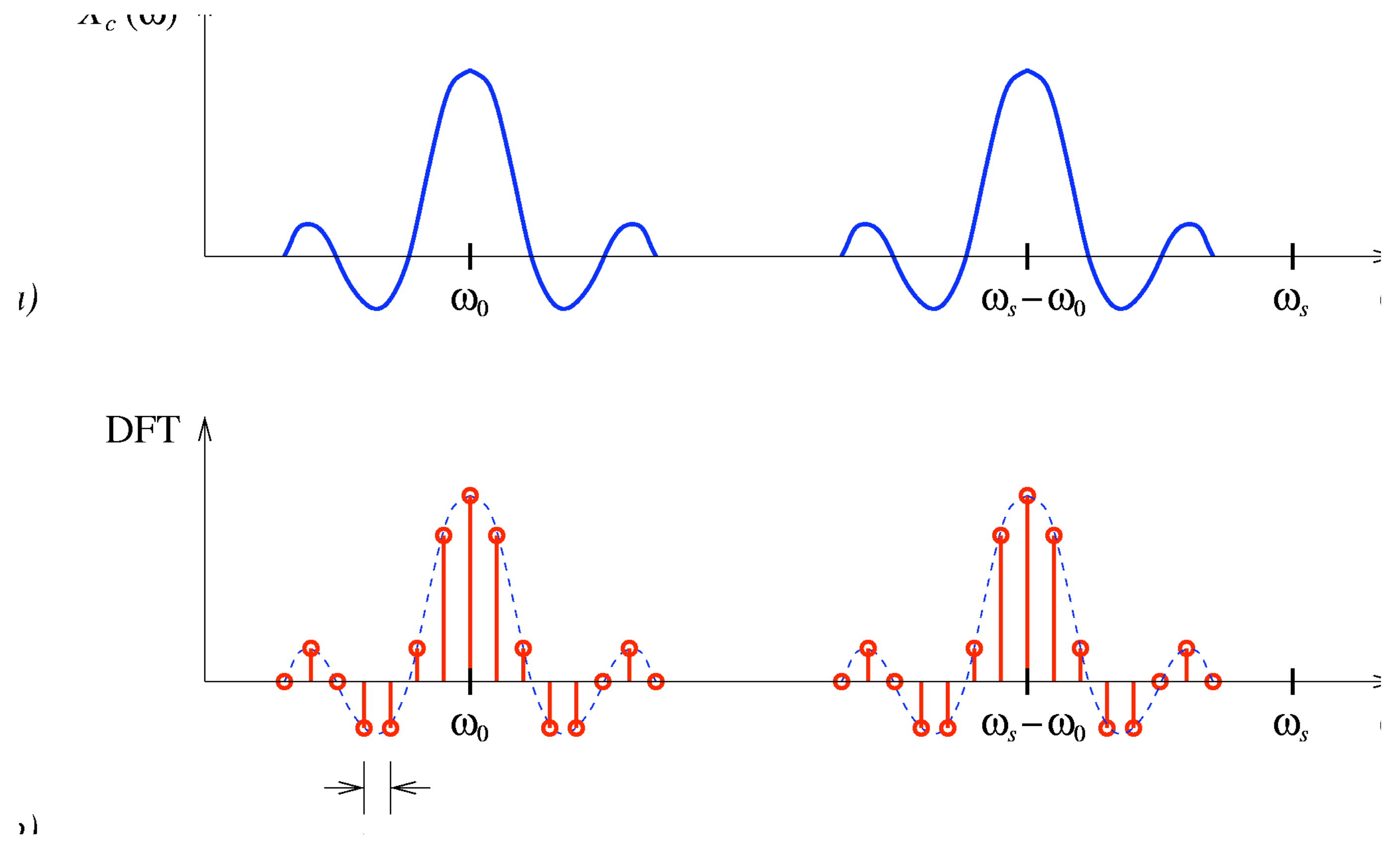


Fig 9.2

Inverse DFT

- The *inverse* DFT (IDFT) is defined as:

$$\boxed{x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\exp\left(\frac{jnk2\pi}{N}\right) \quad 0 \le k, n \le N-1}$$

- Processing identical to DFT, except sign of exp() is reversed
- The IDFT is particularly important for efficient (fast) convolution algorithms

DFT Symmetries

- For real *x(n)*, *X(-k)* can be found to be:

$$X(-k) = \sum_{n=0}^{N-1} x(n)\exp\left(\frac{-jn(-k)2\pi}{N}\right)$$

$$= \sum_{n=0}^{N-1} x(n)\exp\left(\frac{jnk2\pi}{N}\right) = X^*(k)$$

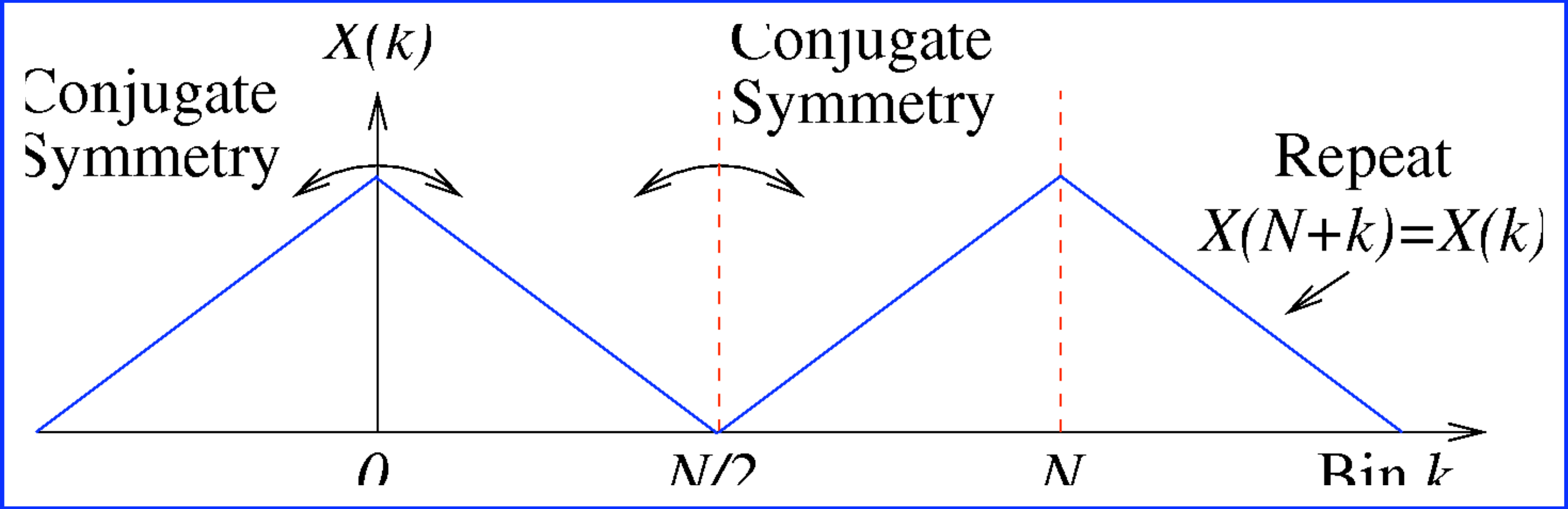
- Similarly for *X(N-k)*: $X(N-k) = X^*(k)$

- For *X(N+k)*, we find that:

$$X(N+k)=\sum_{n=0}^{N-1}\exp\left(\frac{-jn(N+k)2\pi}{N}\right)$$

$$=\sum_{n=0}^{N-1}x(n)\exp\left(\frac{-jnk2\pi}{N}\right)=X(k)$$

- We can summarise symmetries as:



Magnitude vs Phase

- The shift properties imply that:
  - Magnitude DFT of a signal is the same, regardless of its precise timing
  - The phase information in the DFT represents the timing information

Twiddle Factors

- Shorthand for exp() functions:

$$W_N^k = \exp\left(\frac{-jk2\pi}{N}\right)$$

- Twiddles for *N=8*:

$$W_8^{-8} = W_8^8 = W_8^0 = 1,$$

$$W_8^6 = -W_8^2 = j, \text{etc}$$

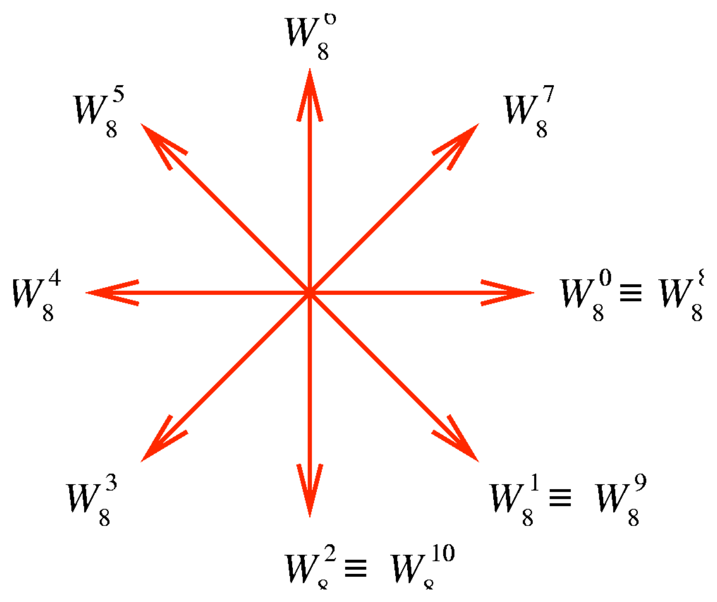


Fig 9.3

DFT Matrix Equation

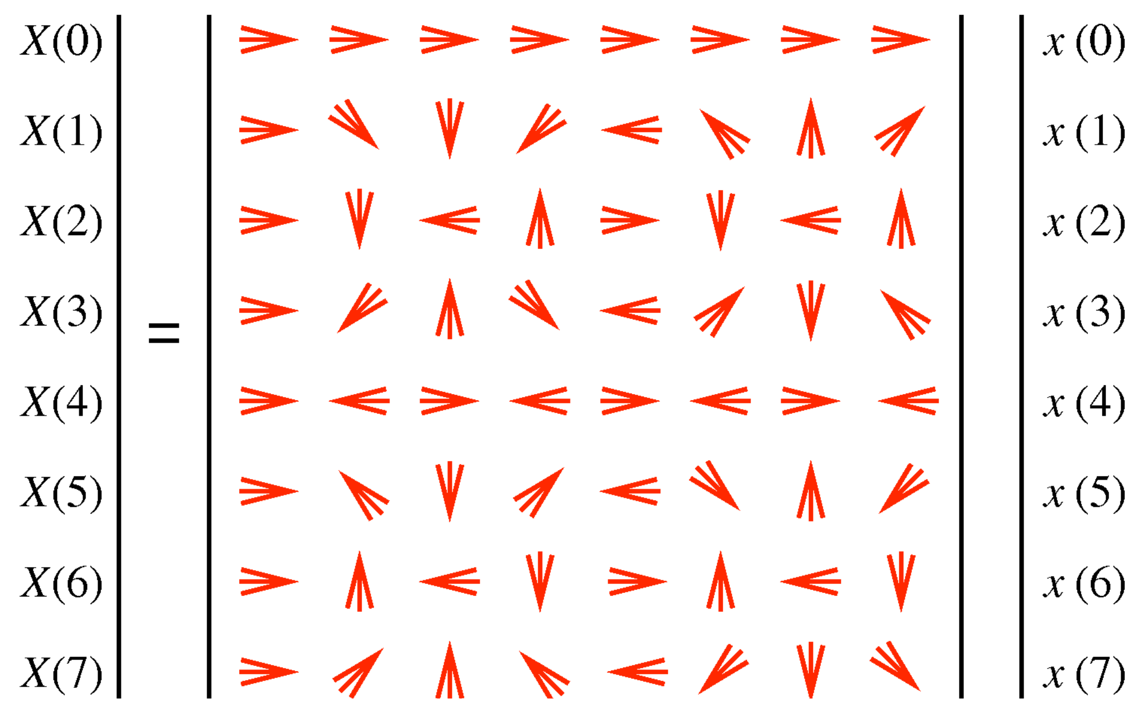


Fig 9.4

- First line of DFT matrix is dc, second is one cycle in *N* samples and so on

DFT Calculation

- An input signal $\exp(j\omega_f n\Delta t)$ rotates anti-clockwise in the complex plane
- The DFT matrix rows rotate **clockwise** at different frequencies
- A row de-rotates exponential signals at that frequency, giving a large bin value

DFT Calculation Example

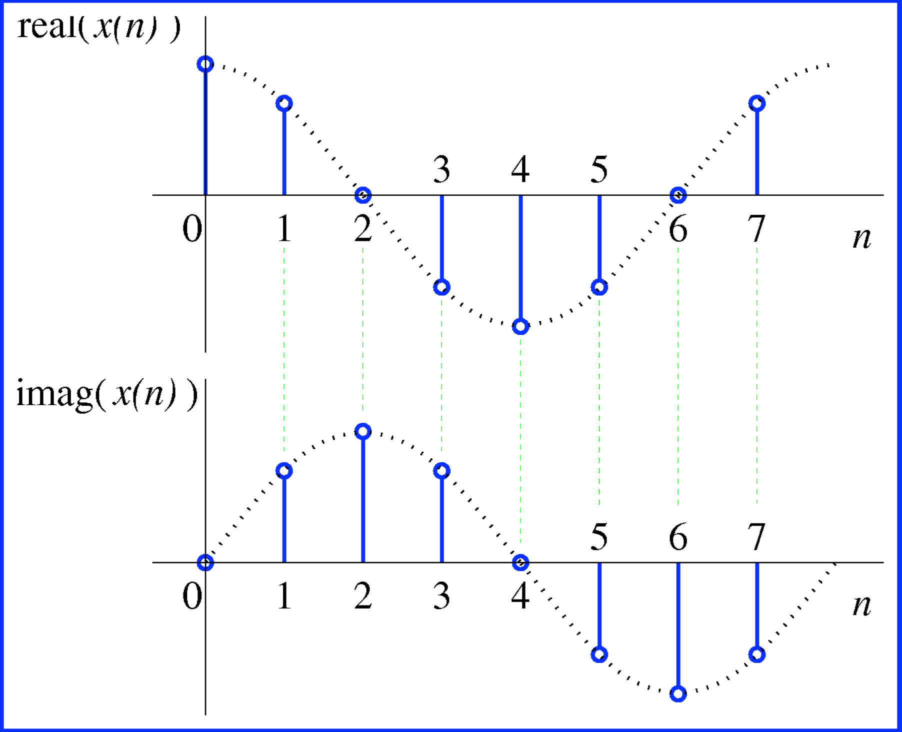
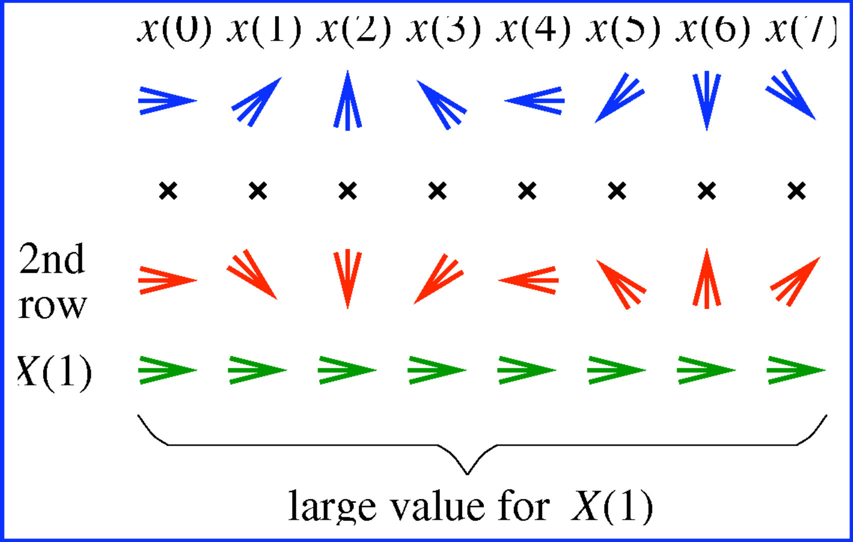


Fig9.5

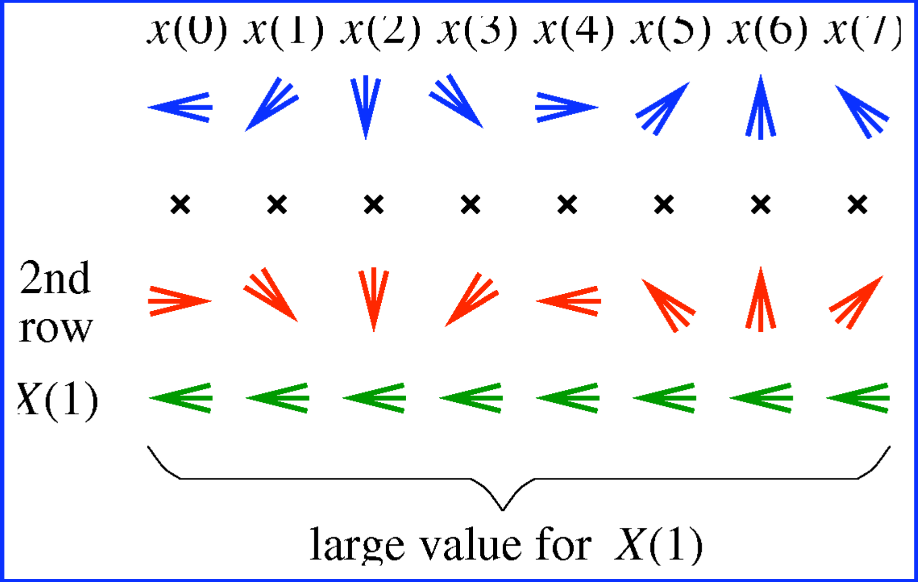


Signal x(n)

DFT Calculation

Phase Offset DFT Example

- Add 180° phase shift to input signal:



- Magnitude of X(1) unchanged, but added 180° phase shift

Summary

- Fourier transform to discrete Fourier transform
- Properties of the DFT
- DFT evaluation