THE DISCRETE FOURIER TRANSFORM (DFT) LECTURE 2

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Chapter 9 of Textbook -- part II

(original material from John Thompson)

Summary

- DFT computation
- Analogies for the DFT
- Leakage and the use of windows

DFT Matrix Equation

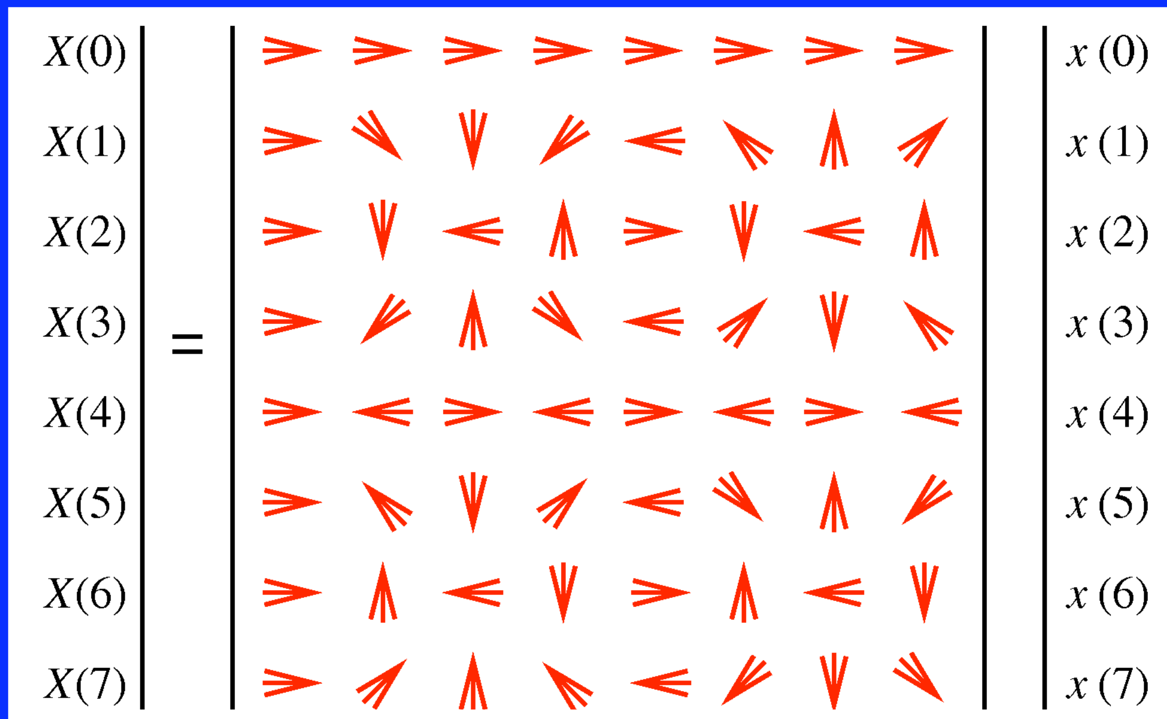


Fig 9.4

- First line of DFT matrix is dc, second is one cycle in *N* samples and so on

DFT Calculation

- An input signal exp($j\omega_f n\Delta t$) rotates anti-clockwise in the complex plane
- The DFT matrix rows rotate **clockwise** at different frequencies
- Each row de-rotates any sine waves at that frequency, giving a large bin value

DFT Calculation Example

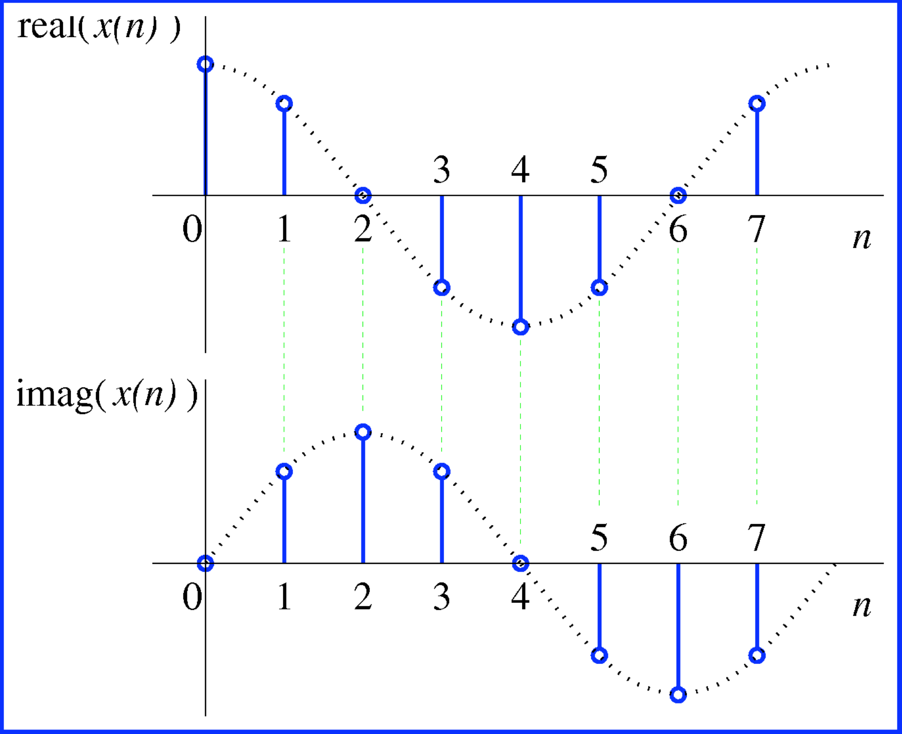
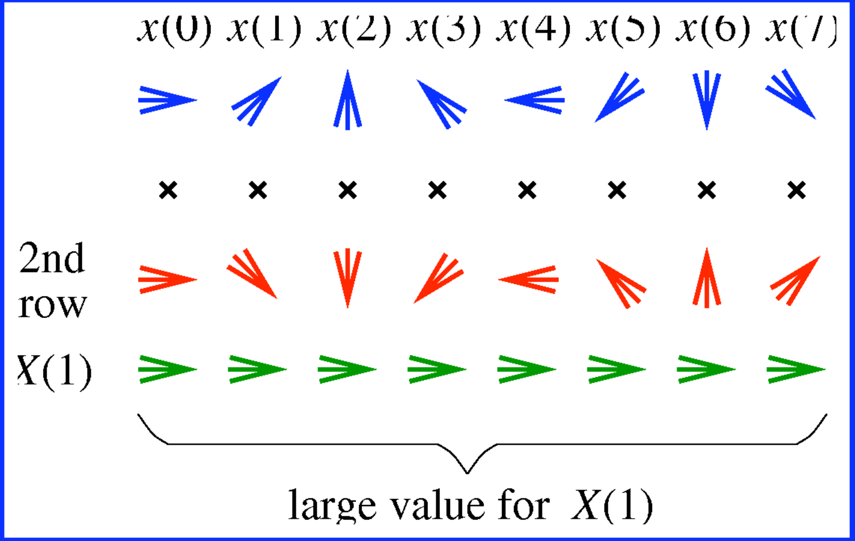


Fig9.5

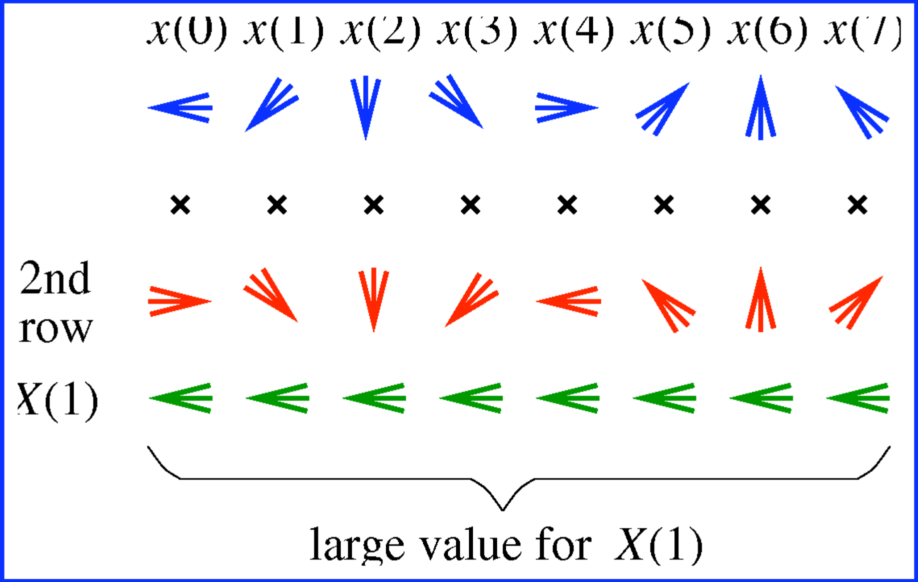


Signal *x(n)*

DFT Calculation

Phase Offset DFT Example

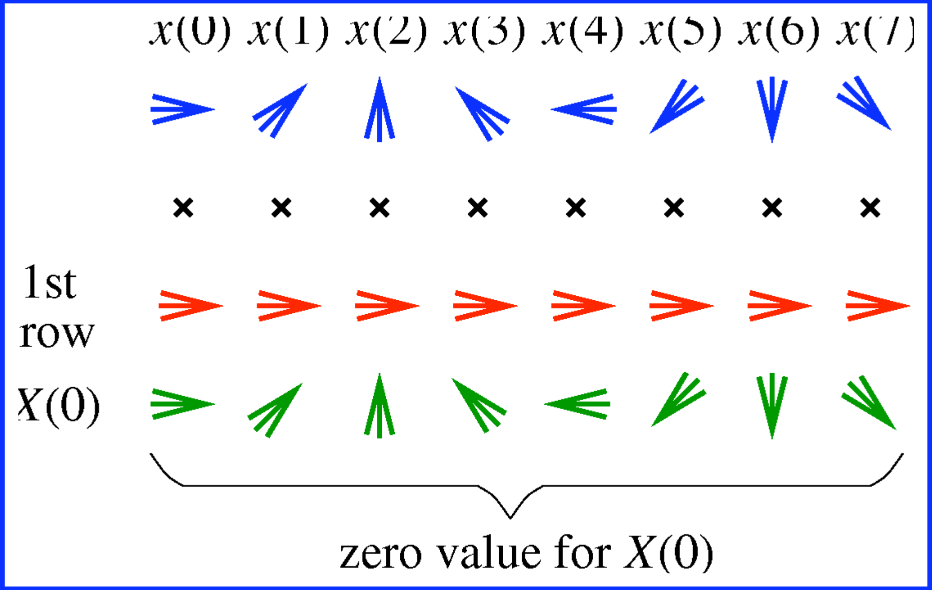
- Add 180° phase shift to input signal:



- Magnitude of X(1) unchanged, but added 180° phase shift

DFT Calculation Example - Bin 0

- Evaluation of *X(0):*



- Product terms cancel each other as they are summed to form *X(0)*

DFT Output

- For our test signal, the DFT output is:

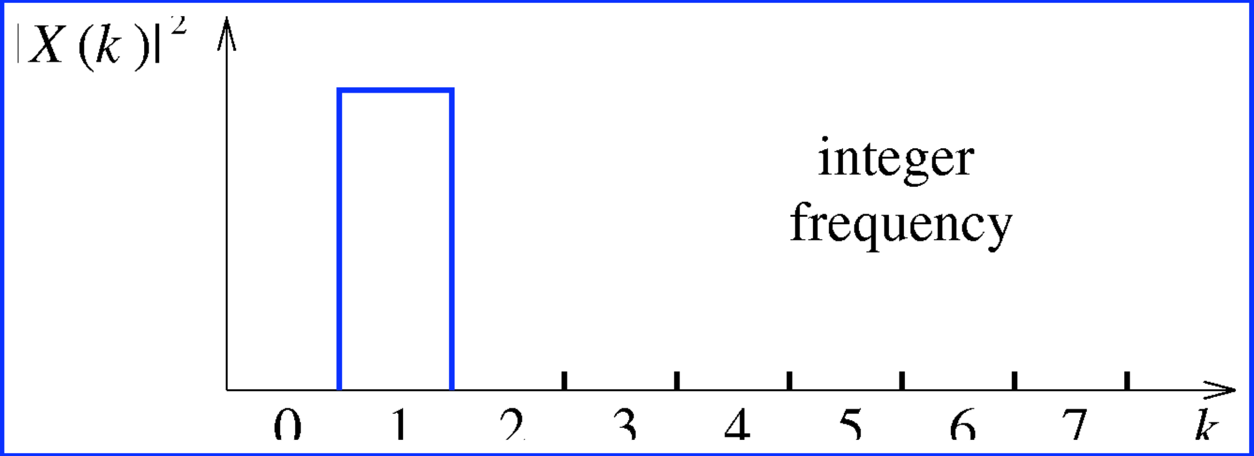


Fig
9.6a

- One non-zero bin at frequency of complex exponential

The DFT as a Filterbank

- The filterbank realisation:

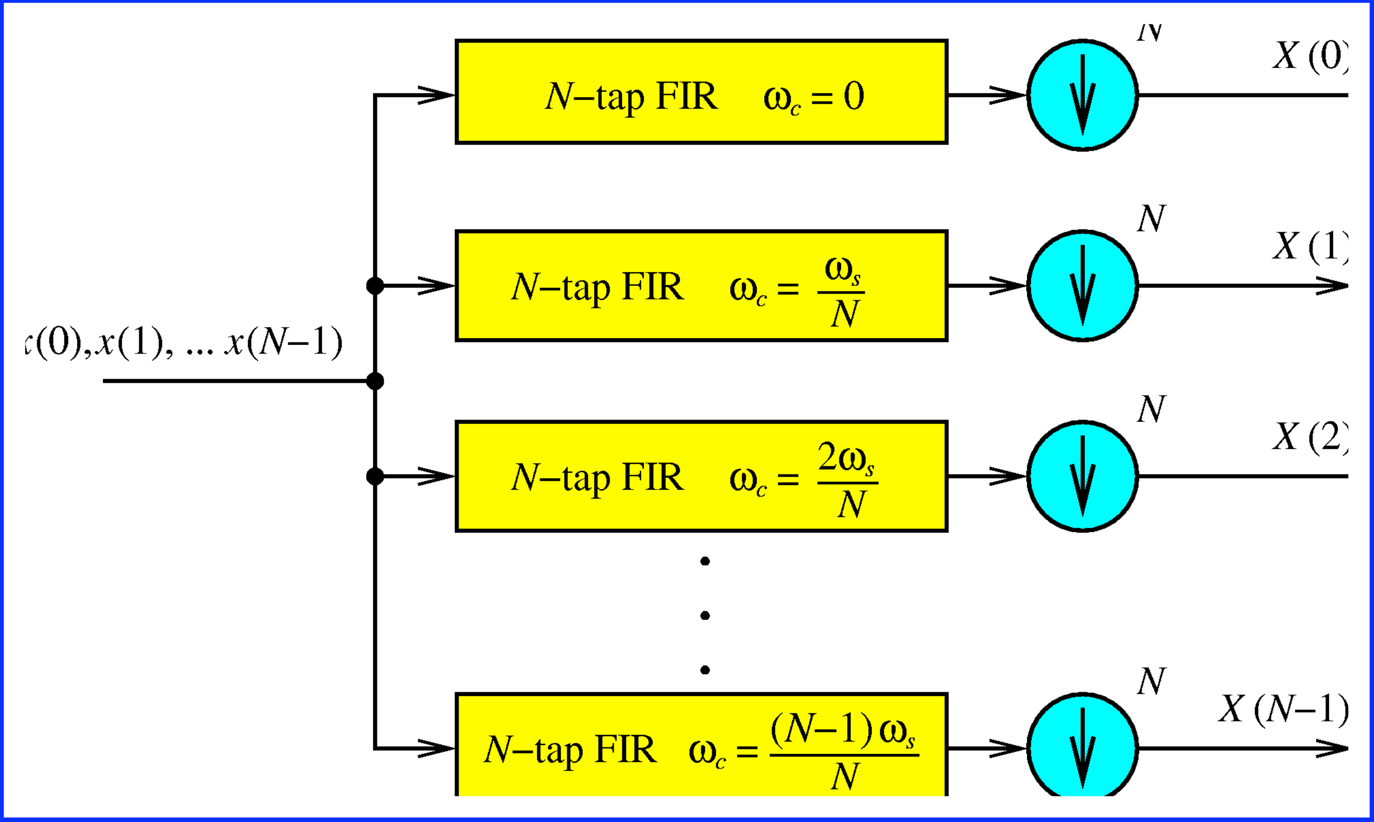


Fig 9.7

- The 8-point DFT thus consists of 8 FIR filters of length 8 taps
- The filters are low-pass or bandpass, with centre frequencies spaced by $\omega_s/8$:

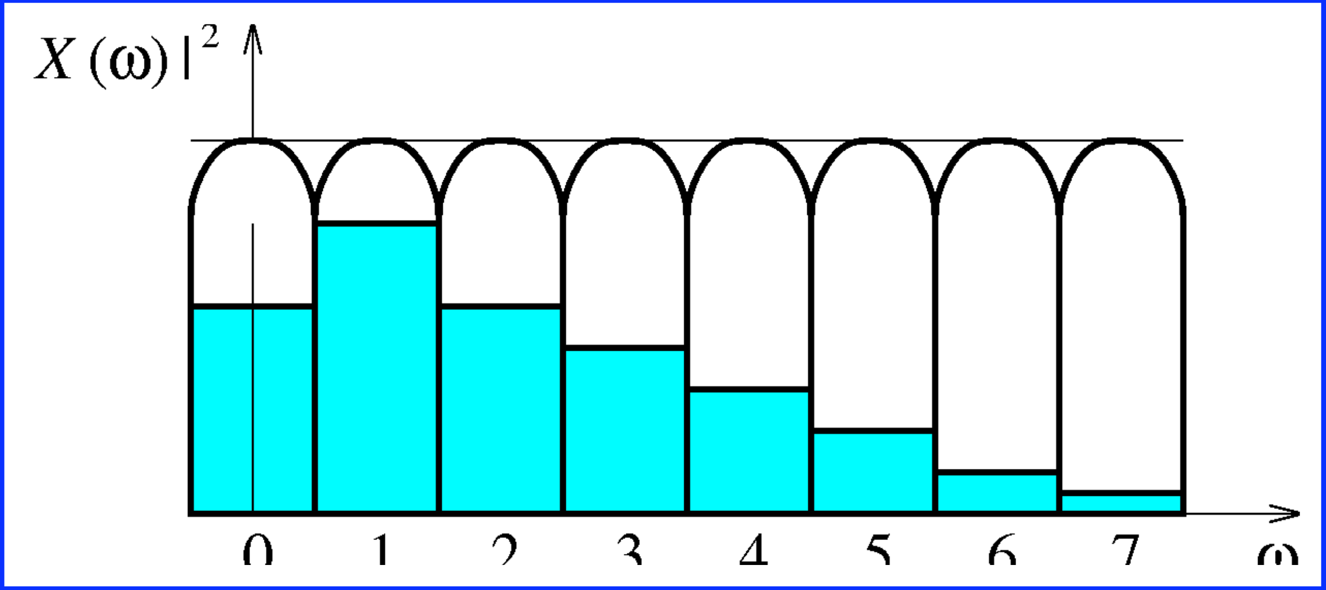
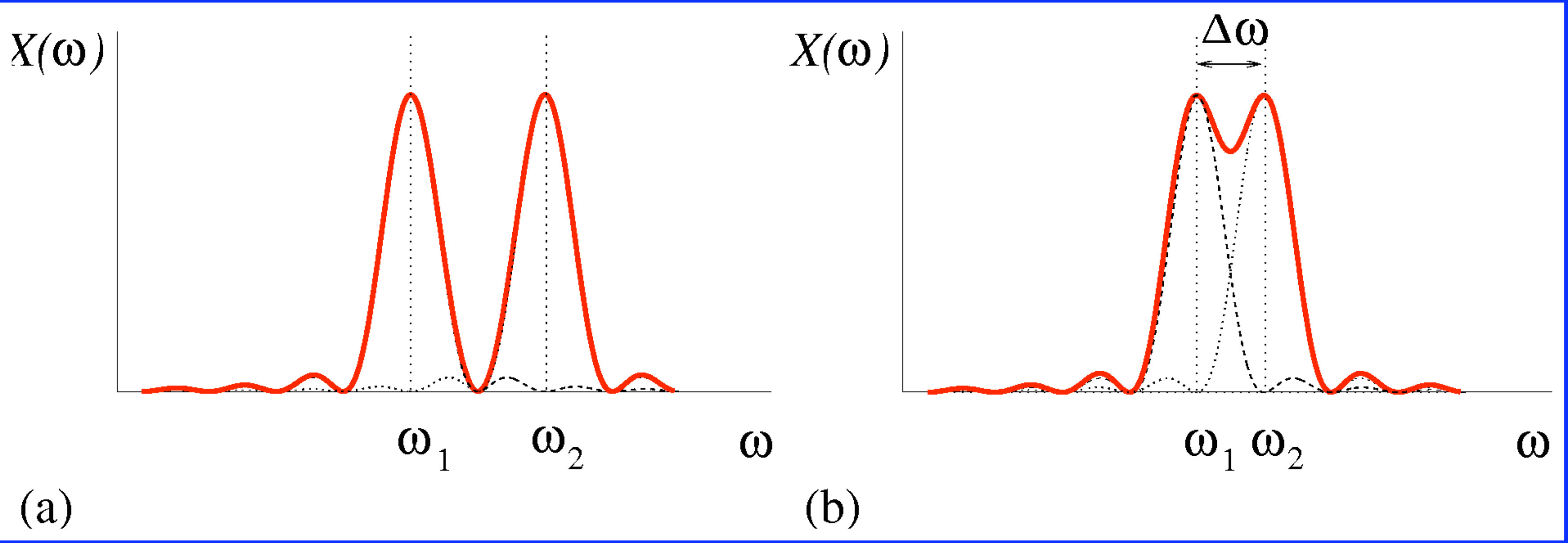


Fig 9.8

DFT Resolution

- Each DFT filter in the filterbank has a sinc(x) shape in frequency
- Resolving two sine waves:
  - Assume sine waves have frequencies equal to those of two consecutive bins
  - DFT yields two peaks in the two bins
- Minimum resolution of an N-point DFT is thus $\Delta\omega = \omega_s / N = 2\pi / N\Delta t$

Visualising Resolution



(a) Two sine waves (b) Minimum resolution sine wave spacing

DFT Leakage

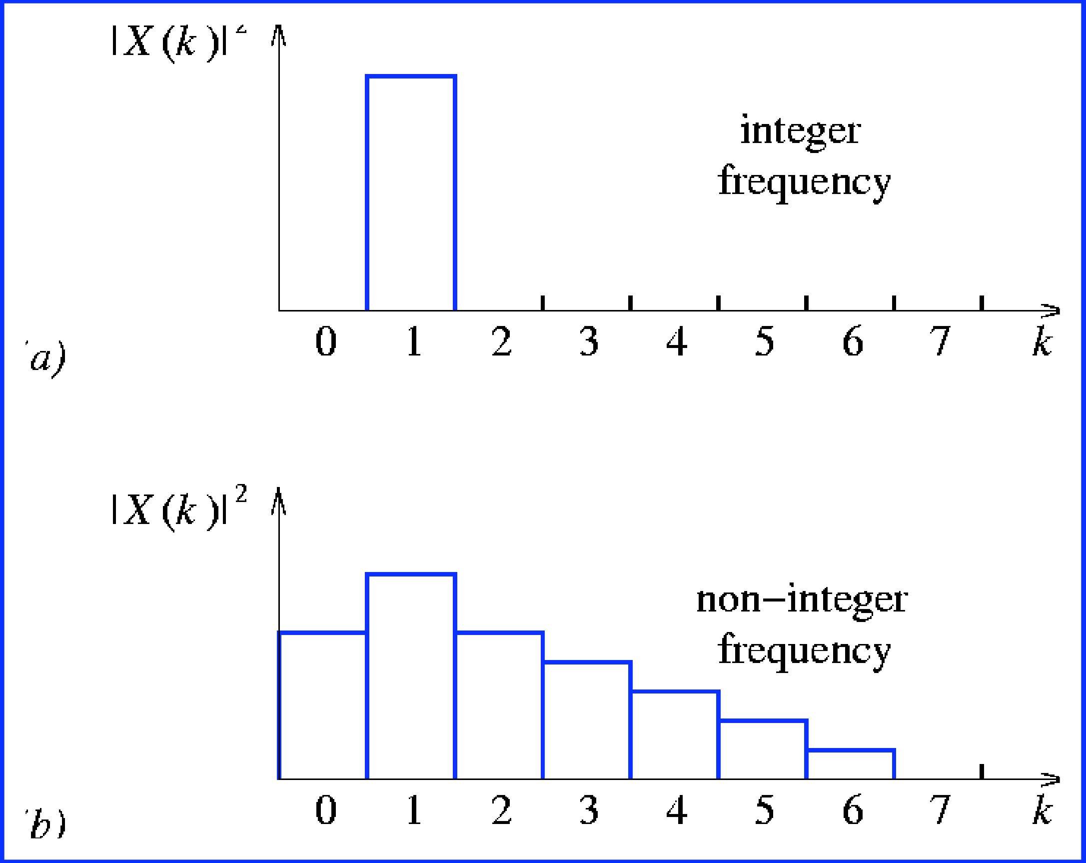


Fig 9.6

- Worst case leakage for $\omega_f=(n+0.5)\Delta\omega$

The Rectangular Window

- Two sine waves set at bin frequencies:

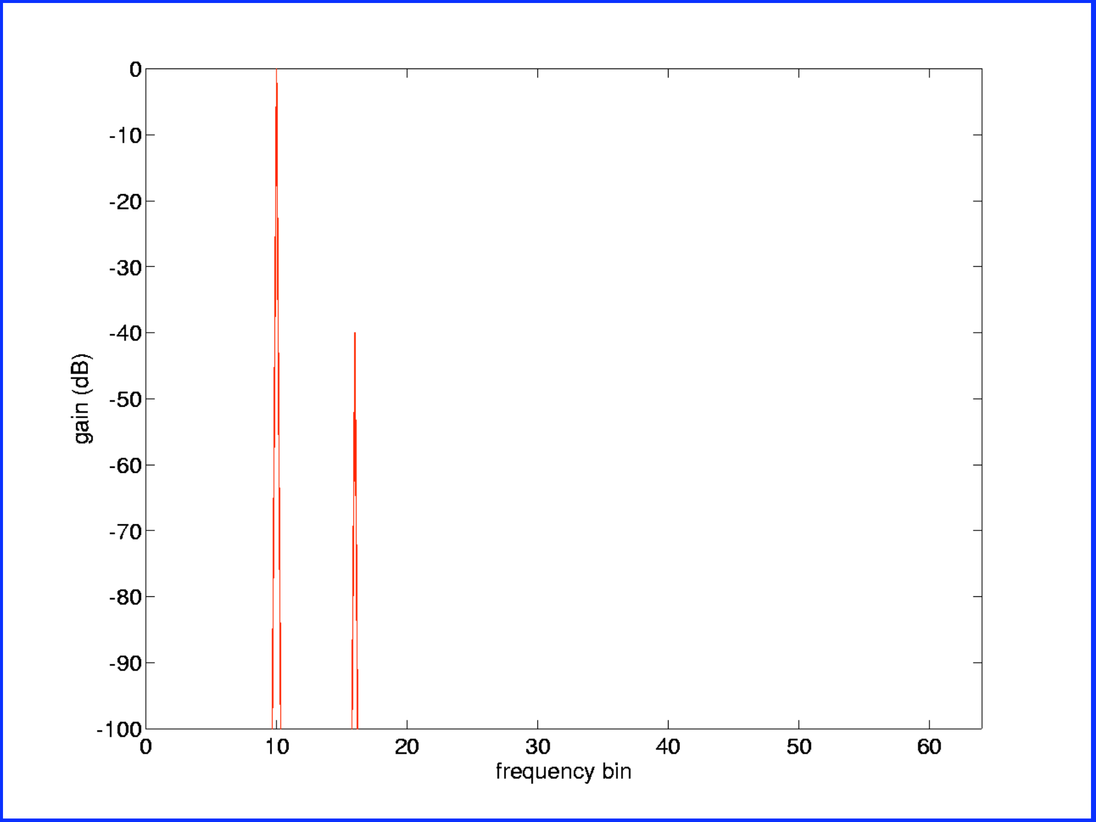


Fig 9.9

- Now set larger sine wave frequency half way between two bins:

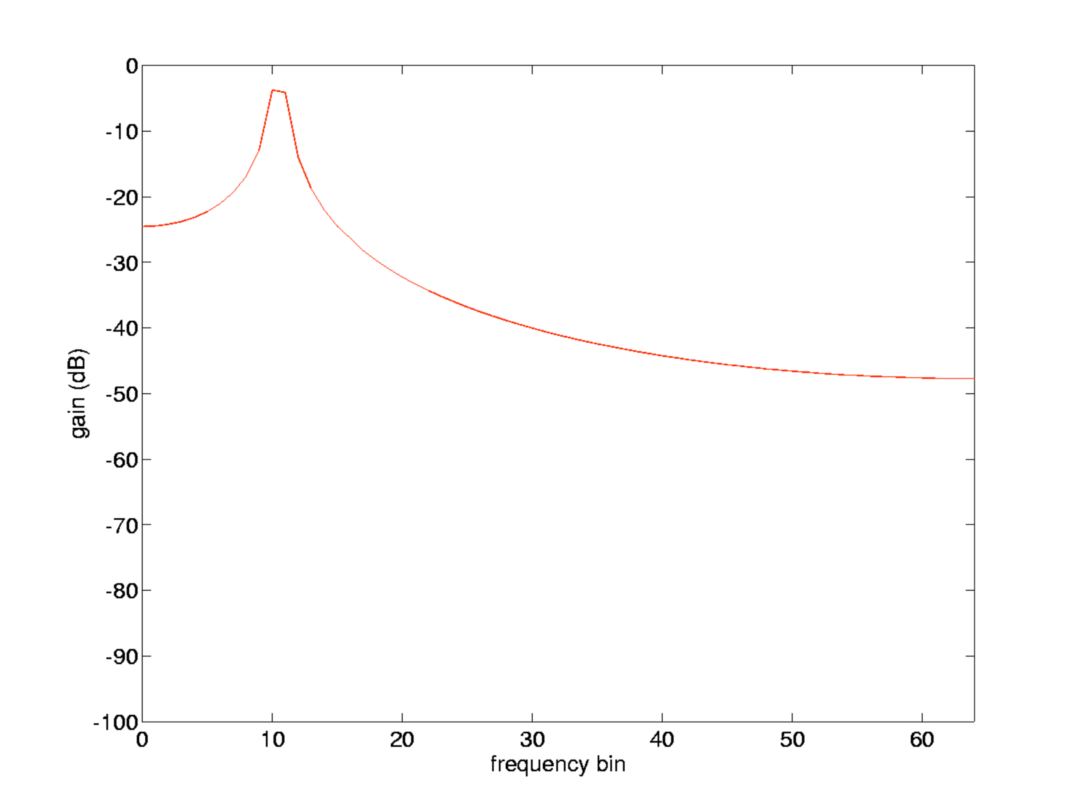


Fig 9.10

- Now we use zero padding to get 1024-point DFT and examine bins 0-200:

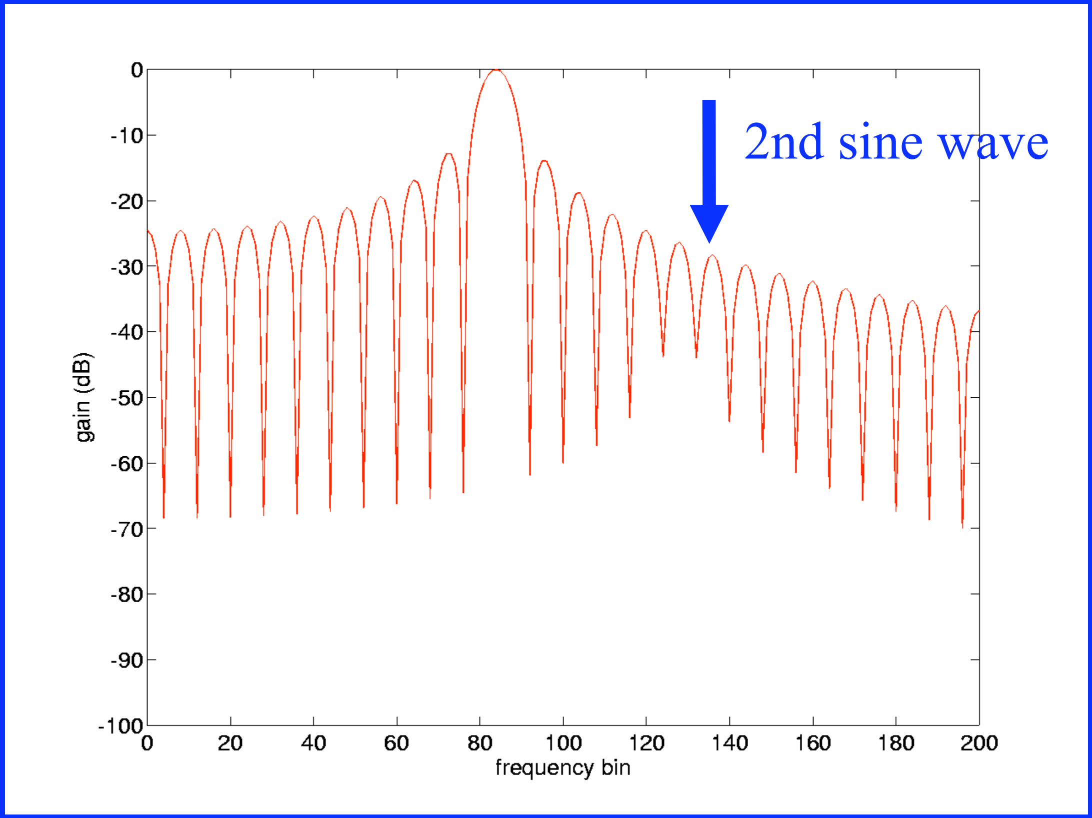


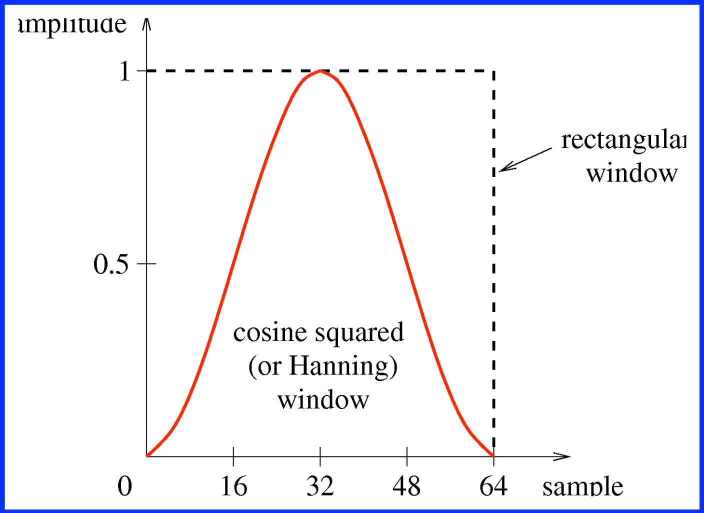
Fig 9.11b

Leakage Effects

- Sine waves lying between bins give leakage into multiple frequency bins
- Leakage caused by sinc(x) frequency response of rectangular window
- In example, second sine wave 40 dB down was lost in the leakage of first sine wave

Windowing Functions

- As in FIR Filter design, multiply signal *x (n)* by a window, <u>before</u> DFT operation
- Hanning window is widely used:



$$w_n = 0.5 - 0.5\cos\left(2n\pi / M\right)$$
$$(n = 0 \ldots M - 1)$$

Fig 9.12

DFT with Hanning window

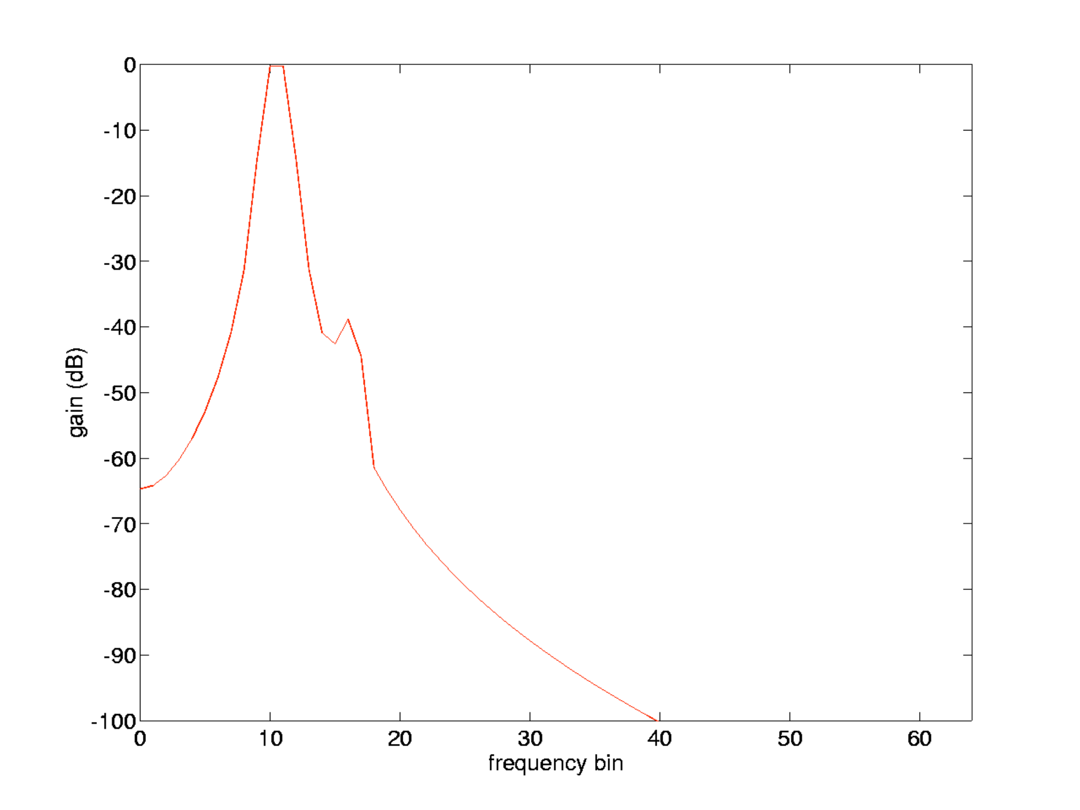


Fig 9.13a

- Both sine waves are resolved (just)
- Main lobes of each peak are wider

Hamming Window

- Can also use the Hamming window:

$$w_n = 0.54 - 0.46\cos(2n\pi/M), \quad n = 0 \ldots M-1$$

- Hamming vs Hanning windows:
  - First sidelobe reduced from -32 dB (Hanning) to -43 dB (Hamming)
  - But decay of sidelobe levels is slower for Hamming than for Hanning

DFT with Hamming Window

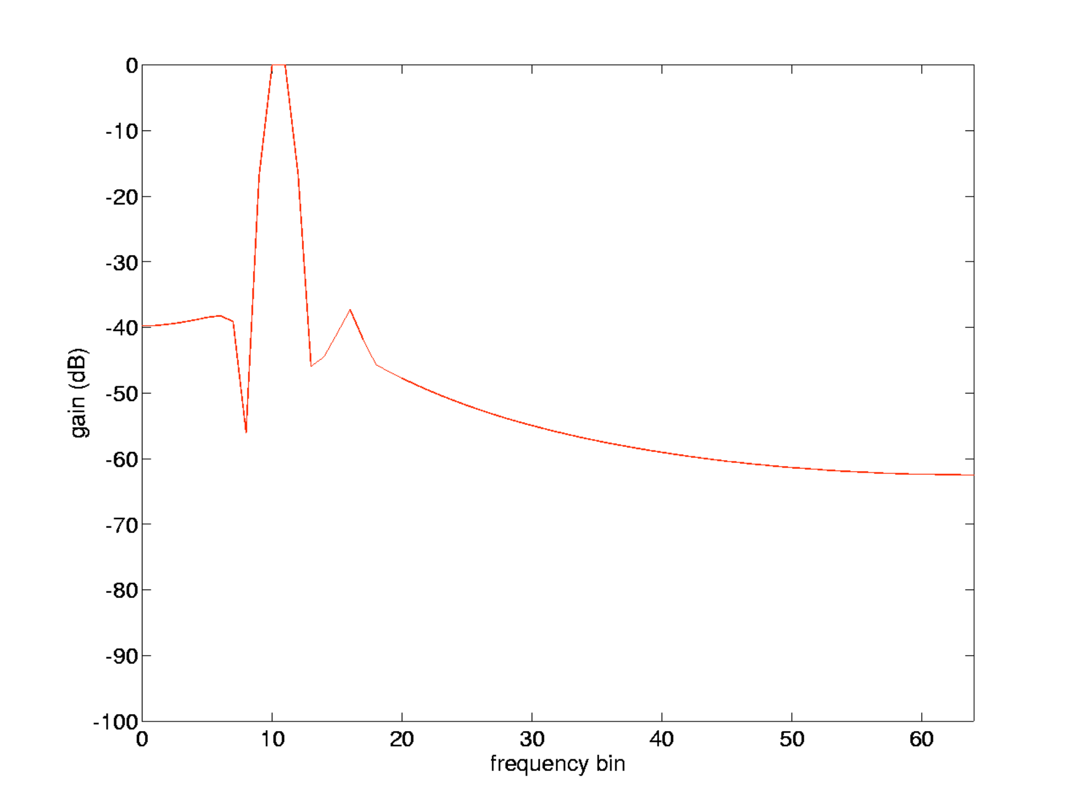


Fig 9.13b

- Second sine wave is barely resolved

Dolph-Chebychev Window

- Window equation is:

$$w_n = \frac{N-1}{N-n-1}\sum_{k=0}^{M}\binom{n-1}{k}\binom{N-n-1}{k+1}\beta^{k+1}, \quad n \neq 0, N-1$$

- Where: $M=n-1 (n<=N/2)$, $M=N-n-2 (n<=N/2)$
- w_n has value 1 for $n=0,N-1$
- β is defined as: $\beta = \cosh(\cosh^{-1}(10^{\alpha})/N)$

Dolph-Chebychev Windows

- These windows provide constant sidelobe attenuation
- The value α is the log of sidelobe amplitude level, eg -2 → -40 dB
- Main lobe has the minimum width possible for the given leakage level

DFT with Dolph-Chebychev Window (α=3)

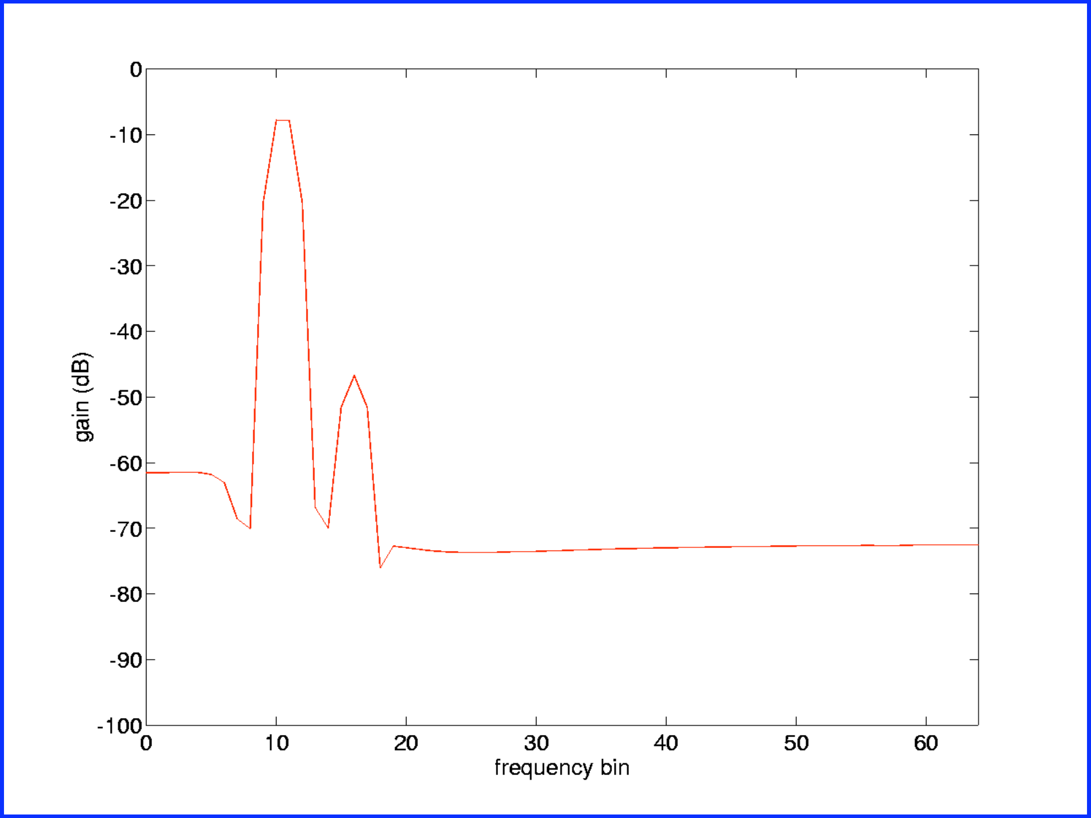


Fig 9.14a

- Sine wave peaks are clearer but main lobes are now wider too

Window Comparisons (Table 9.1)

Window	Main lobe width (Hz)	Sidelobe level	Roll-off rate
Rectang-ular	$\frac{0.89}{N\Delta t}$	-13 dB	6dB/dec
Hanning	$\frac{1.4}{N\Delta t}$	-32 dB	18dB/dec
Hamming	$\frac{1.3}{N\Delta t}$	-43 dB	6dB/dec
Dolph-Chebychev	$\frac{1.44}{N\Delta t}$	-60 dB	0dB/dec

Summary

- Fourier transform to discrete Fourier transform
- Properties of the DFT
- DFT evaluation