DIGITAL SIGNAL PROCESSING (DSP)

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CALENDAR OF DSP COURSE

- Introduction
- Time-Domain Analysis of Discrete Signals and Systems
- Frequency-Domain Analysis Using Fourier and z-Transforms
- Fast Fourier Transform (FFT) Processing
- Design of Nonrecursive and Recursive Digital Filters
- Random Signal Analysis
- DSP Processors
- Adaptive Filtering and Adaptive Signal Processing
- Spectral Analysis and Signal Compression

COURSE TEXTBOOK

- Alan V. Oppenheim and Ronald W. Schafer

  Discrete-Time Signal Processing

  1st (1989) or 2nd (1999) Edition

  Prentice Hall, NJ

  ISBN 0-13-754920-2

  http://www.prenhall.com

1 INTRODUCTION TO DISCRETE SIGNALS

1.1 The scope of DSP

Examples of analogue signals appearing in nature:

- electrical signals: voltages, currents, fields
- acoustic signals: mechanical vibrations, sound waves
- mechanical signals: displacements, velocities, forces, moments

Analogue processing may include the following operations:

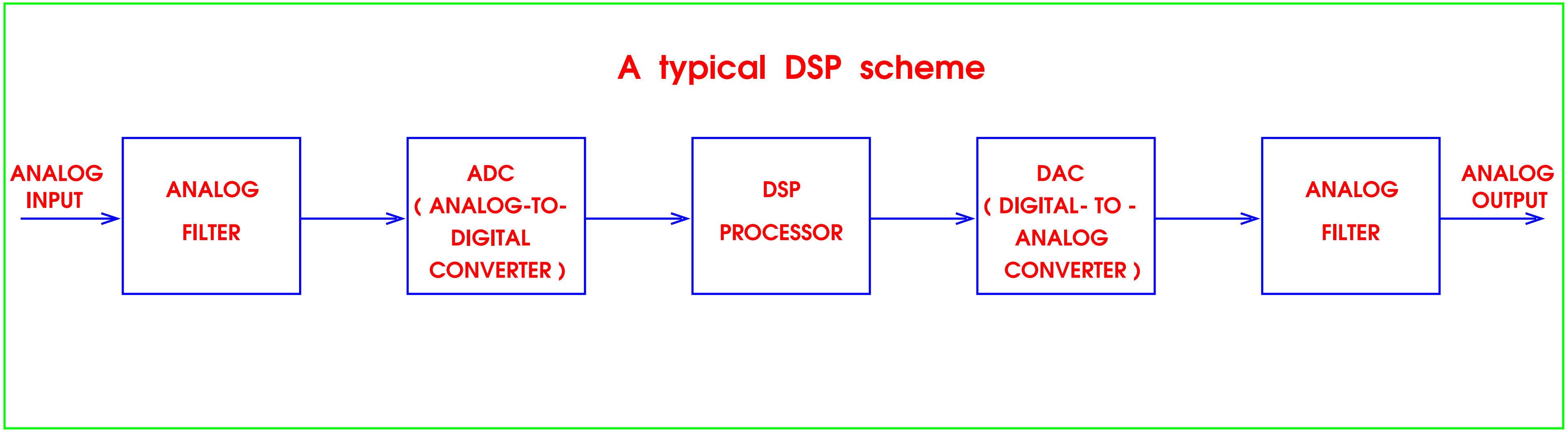
- linear: amplification, filtering, integration, differentiation
- nonlinear: squaring, rectification, inversion

Limitations of practical analogue processing:

- restricted accuracy (e.g., component variations with time, temperature, etc.)
- restricted dynamic range
- sensitivity to noise
- inflexibility to alter or adjust the processing functions
- problems in implementing accurate nonlinear and time-synchronized operations
- high cost of data storage and transmission
- limited speed of operation

DSP operations:

- converting analogue signals into a digital (usually binary) sequence
- performing all signal processing operations in the digital form
- if necessary, converting the digital information back to analogue signal



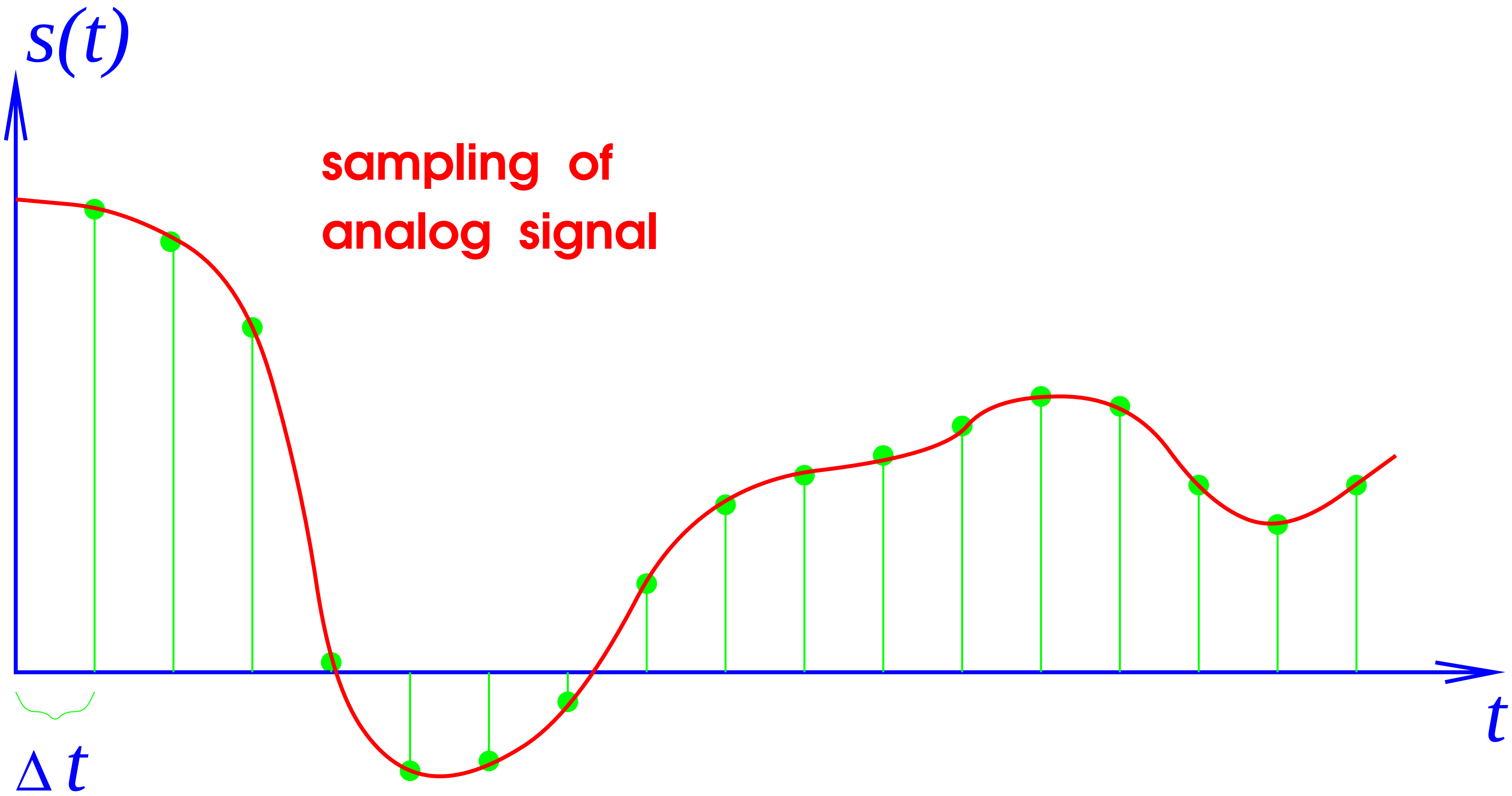
Advantages of DSP:

- digital data storage and transmission is much more effective than in the analogue form
- flexibility: processing functions can be altered or adjusted
- possibility of implementing much more complicated processing functions than in analogue devices
- efficient implementation of fast algorithms and matrix-based processing
- speed of digital operation tends to grow rapidly with the years of technical progress
- a very high accuracy and reliability is possible to achieve
- dynamic range can be increased
- signal multiplexing: simultaneous (parallel) processing

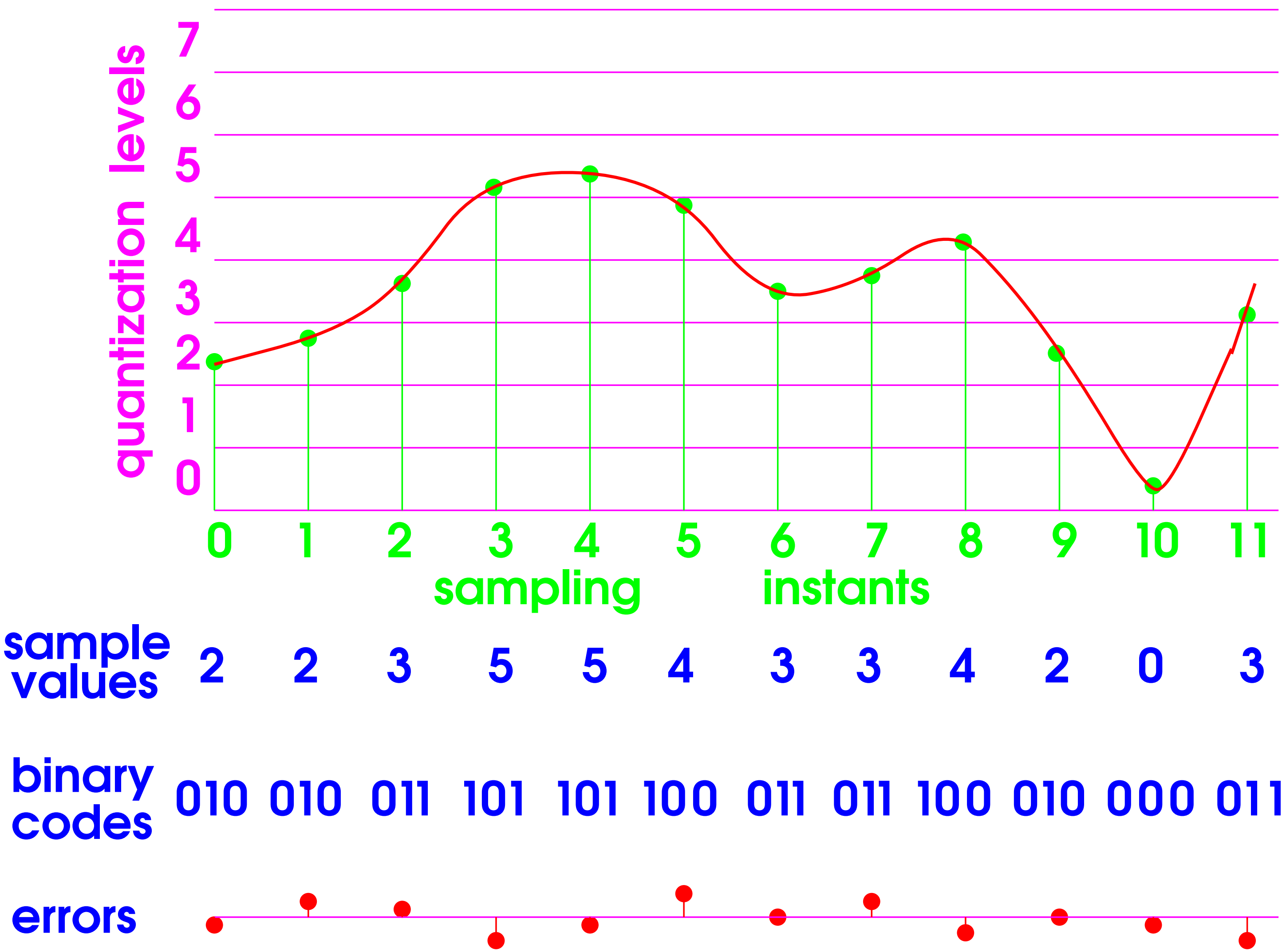
Some application areas of DSP:

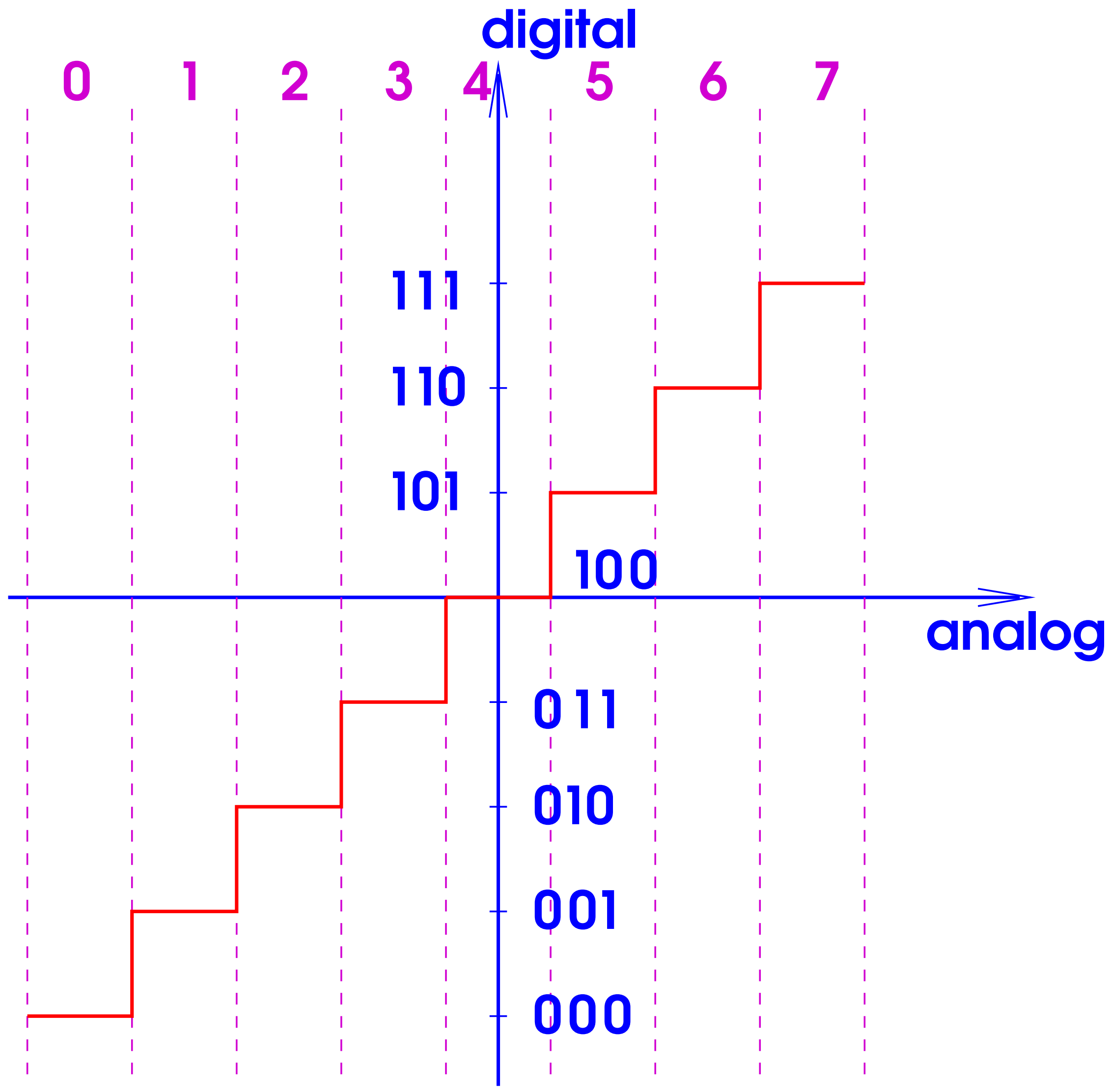
- Music: recording, playback, mixing, synthesis, storage (e.g. CD-players)
- Speech: recognition, synthesis (e.g. automatic speakers)
- Communications and multimedia: signal generation, transmission, modulation and compression, data protection via error-correcting signal coding, (e.g. digital modems, TV and telephony, computers, videoconferencing and Internet)
- Radar: filtering, detection, feature extraction, localization, tracking, identification (e.g. air-traffic control)
- Image processing: 2-D filtering, enhancement, compression, pattern recognition (e.g. satellite images)
- Biomedicine: diagnosis, patient monitoring, preventive care

1.2 Sampling and Analog-to-Digital Conversion



Converting an analog signal into a binary code





Input-output characteristic of 3-bit quantizer

1.3 Basic Types of Digital Signals

Unit-step function:

$$u(n) = \begin{cases} 0\,, & n < 0 \\ 1\,, & n \geq 0 \end{cases}$$

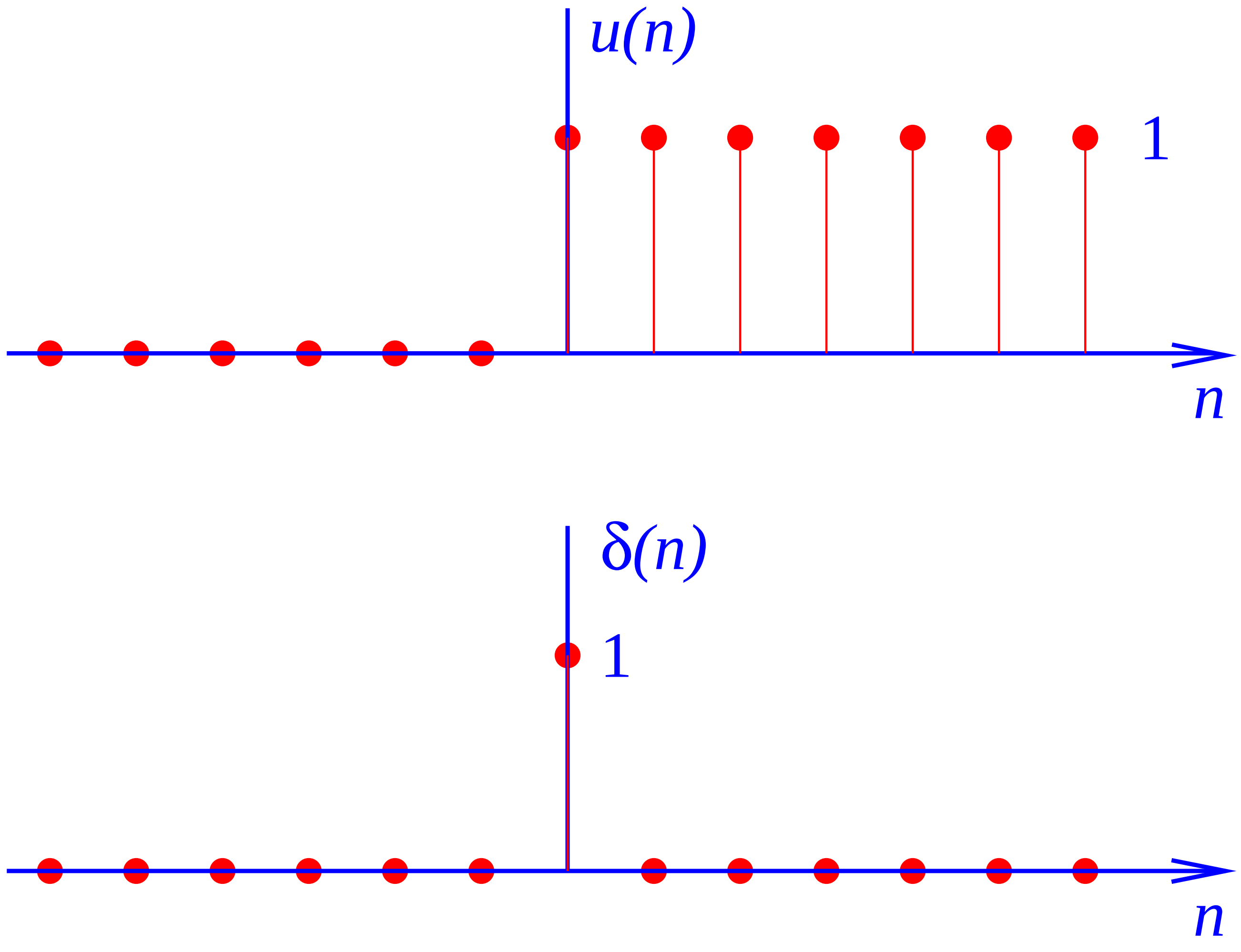
Unit-impulse function:

$$\delta(n) = \begin{cases} 1\,, & n = 0 \\ 0\,, & n \neq 0 \end{cases}$$

$$u(n) = \sum_{m=-\infty}^{n} \delta(m) \qquad \text{integration}$$

$$\delta(n) = u(n) - u(n-1) \qquad \text{differentiation}$$

The unit-step and unit-impulse functions



Periodic signals:

$$x(t) = x(t+T) \quad \Big| \quad x(n) = x(n+N)$$

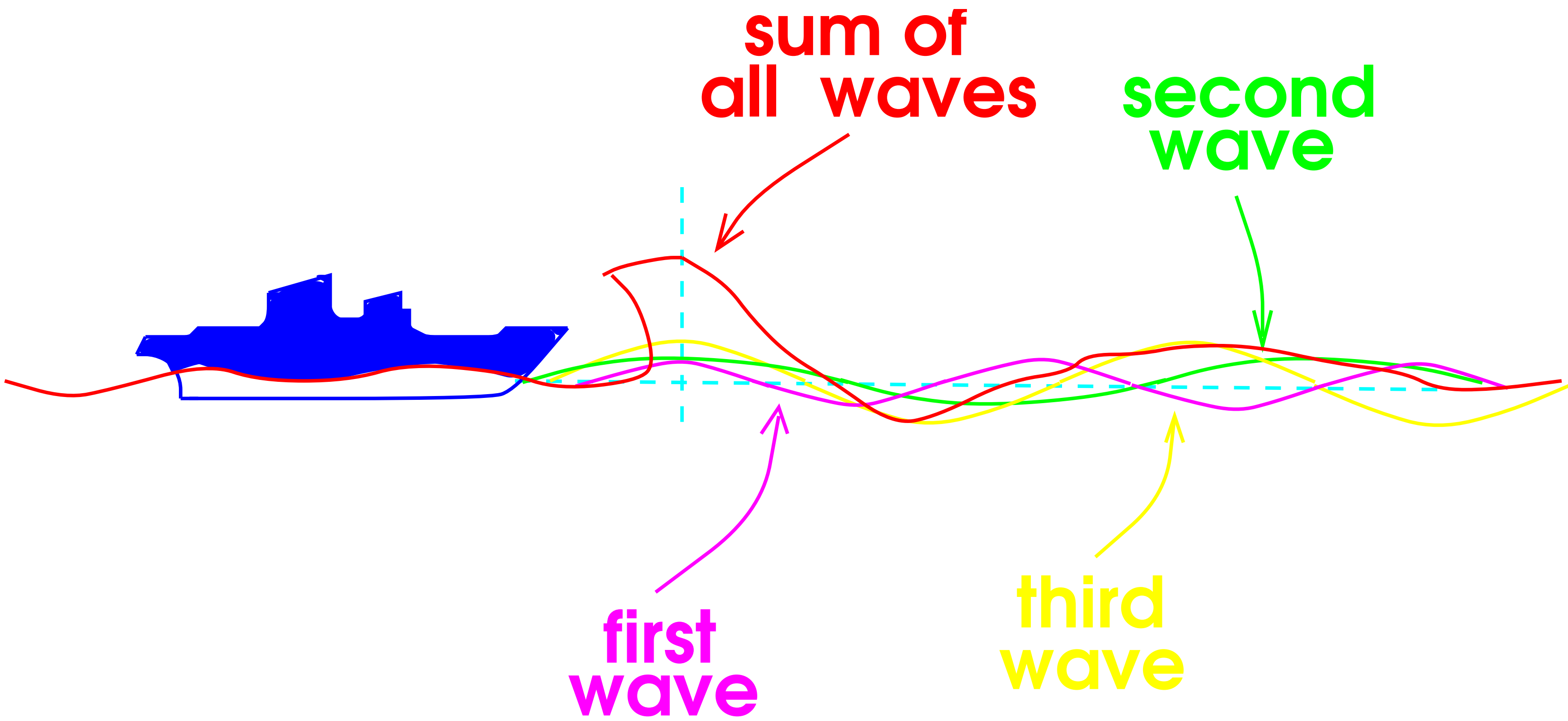
Finite-energy signals:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty \quad \Big| \quad E = \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

Finite-power signals:

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt < \infty \quad \Big| \quad P = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} |x(n)|^2 < \infty$$

Harmonic signals: an example of importance



Complex exponentials (cisoids):

$$x(t) = A\,e^{j(\omega t+\phi)} \qquad \Big| \qquad x(n) = A\,e^{j(\omega n+\phi)}$$

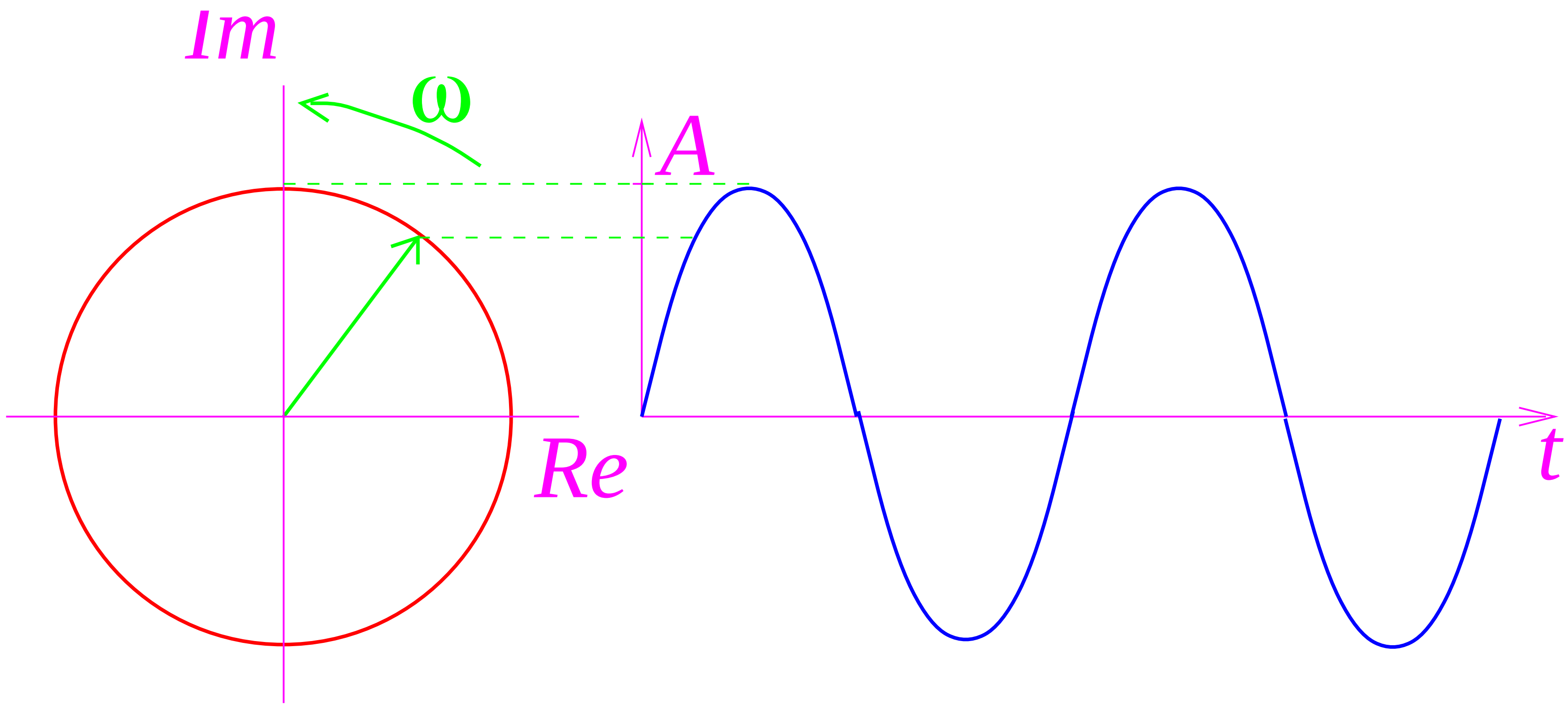
Sinusoids:

$$x(t) = A\,\sin(\omega t+\phi) \qquad \Big| \qquad x(n) = A\,\sin(\omega n+\phi)$$

$$\omega \;\longrightarrow\; \omega\,\Delta t\,,\;\; \text{i.e.} \quad \omega_{\text{discrete}} = \omega_{\text{analog}}\,\Delta t$$

$$\begin{aligned}
e^{j(\omega n+\phi)} &= \cos(\omega n+\phi) + j\sin(\omega n+\phi) \\
\cos(\omega n+\phi) &= \{e^{j(\omega n+\phi)} + e^{-j(\omega n+\phi)}\}/2 \\
\sin(\omega n+\phi) &= \{e^{j(\omega n+\phi)} - e^{-j(\omega n+\phi)}\}/2j
\end{aligned}$$

A sine wave as the projection of a complex phasor onto the imaginary axis:



Differencies between sampled exponentials and their analog counterparts:

- analog exponentials and (co)sinusoids are periodic with $T = 2\pi/\omega$, discrete sinusoids are not necessarily periodic (although their values lie on a periodic envelope).

  Periodicity condition:

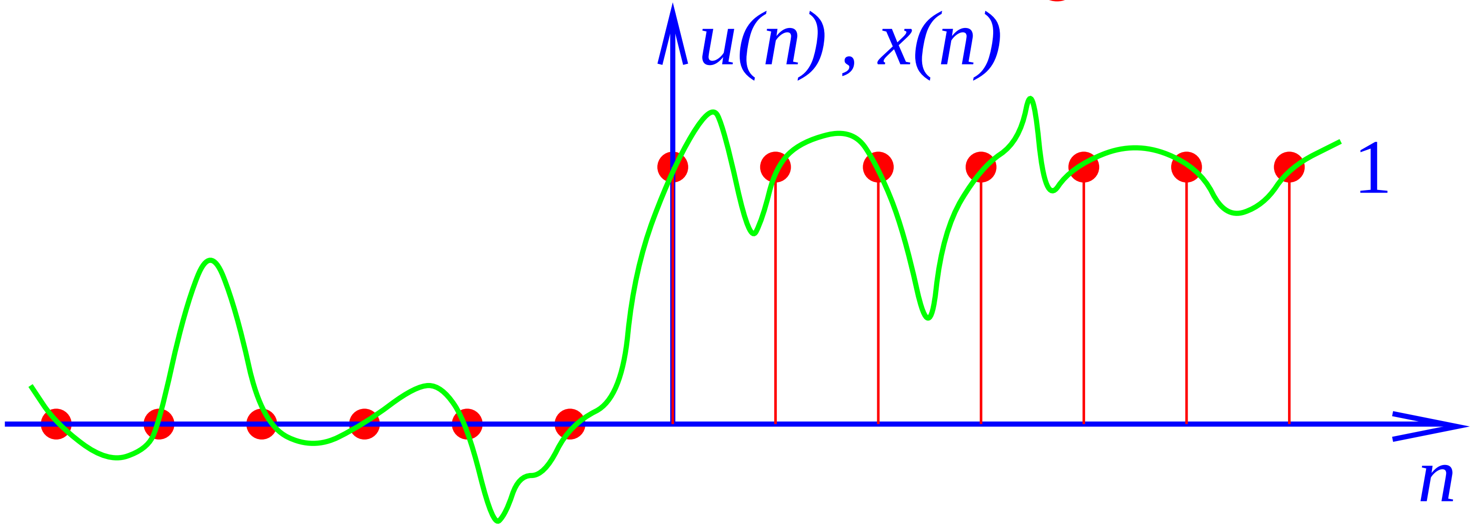
$$x(n) = x(n+N) \quad \Longrightarrow \quad e^{j\omega n} = e^{j\omega(n+N)} \quad \Longrightarrow \quad \exp\{j\omega N\} = 1$$

$$\Longrightarrow \quad \omega = \frac{2\pi m}{N} \quad \text{or} \quad f = \frac{m}{N} \quad (\omega = 2\pi f)$$

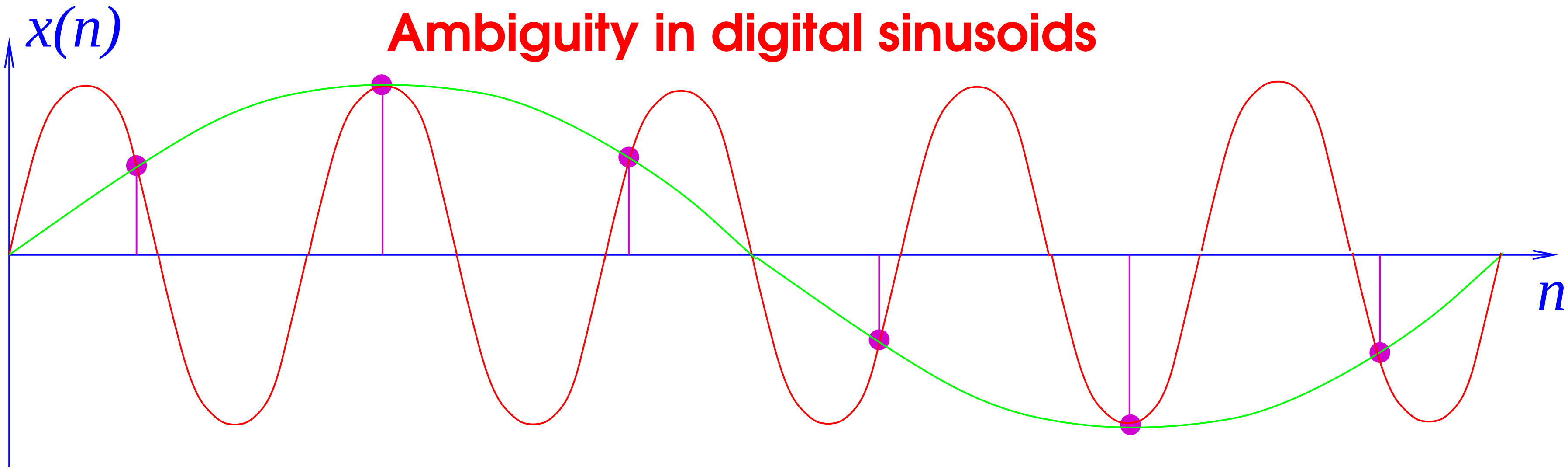
  (this result applies to sines and cosines as well!)

- for sampled exponentials, the frequency ω should be measured in [radians] rather than [radians per second]
- digital signals have ambiguity

The unit-step function and one of many analog signals which can be drawn through its sample points



Ambiguity in digital sinusoids



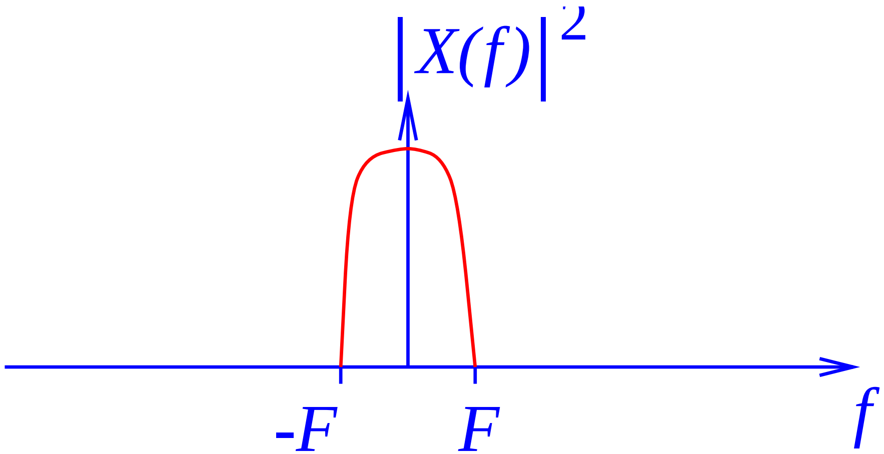
Ambiguity condition for digital sinusoids:

$$\sin(\omega_1 \Delta t) = \sin(\omega_2 \Delta t), \quad \omega_1 \neq \omega_2 \implies$$

$$2\pi f_1 \Delta t = 2\pi f_2 \Delta t + 2\pi m, \qquad m = \ldots, -2, -1, 1, 2, \ldots \implies$$

$$|f_1 - f_2| = m/\Delta t, \quad m = 1, 2, \ldots \qquad (*)$$

Example: lowpass signal (whose spectrum $|X(f)|^2$ is concentrated in the interval $[-F, F]$)



Taking $f_1 = -F$ and $f_2 = F$ in $(*)$, we obtain that there is no ambiguity if the signal is sampled with

$$f_s = \frac{1}{\Delta t} > 2F \qquad (**)$$

where f_s is the so-called sampling frequency.

Remarks:

- The frequency $f_N = 2F$ is referred to as the Nyquist rate
- Digital signal ambiguity is often termed as the aliasing effect
- Equation $(**)$ represents the particular formulation of the Shannon-Nyquist-Kotelnikov Sampling Theorem
- We'll study the differencies between digital and analog signals further using frequency-domain signal analysis

2 TIME-DOMAIN ANALYSIS

2.1 Linear Time-Invariant (LTI) Systems

Definition of a system:

$$y(n) = T\{x(n)\}$$

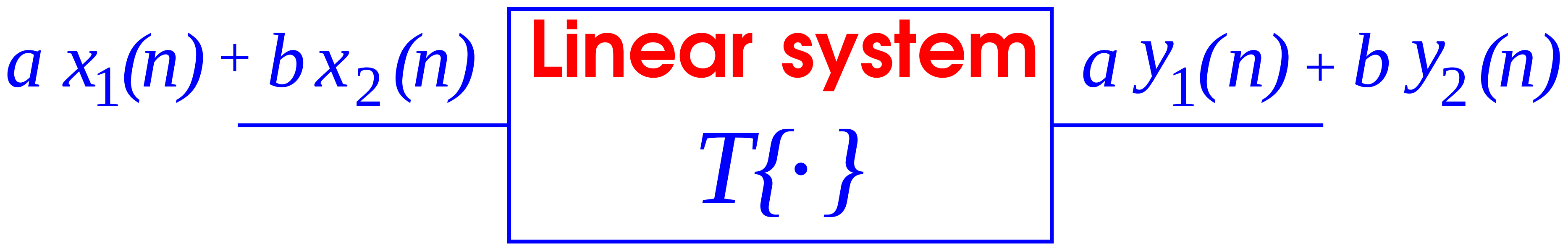
where $T\{\cdot\}$ is an operator that maps an input sequence $x(n)$ into an output sequence $y(n)$.

Linear system: A system (or processor) is linear if it obeys the principle of superposition.

Principle of superposition: If the input of a system contains the sum of multiple signals then the output of this system is the sum of the system responses to each separate signal.

A system is linear if and only if:

$$
\begin{aligned}
T\{ax_1(n) + bx_2(n)\} &= aT\{x_1(n)\} + bT\{x_2(n)\} \\
&= ay_1(n) + by_2(n)
\end{aligned}
$$



Example: Let $y(n) = x^2(n)$ (i.e., $T\{\cdot\} = (\cdot)^2$). Then,

$$\begin{aligned} T\{x_1(n) + x_2(n)\} &= x_1^2(n) + x_2^2(n) + 2x_1(n)x_2(n) \\ &\neq x_1^2(n) + x_2^2(n) \end{aligned}$$

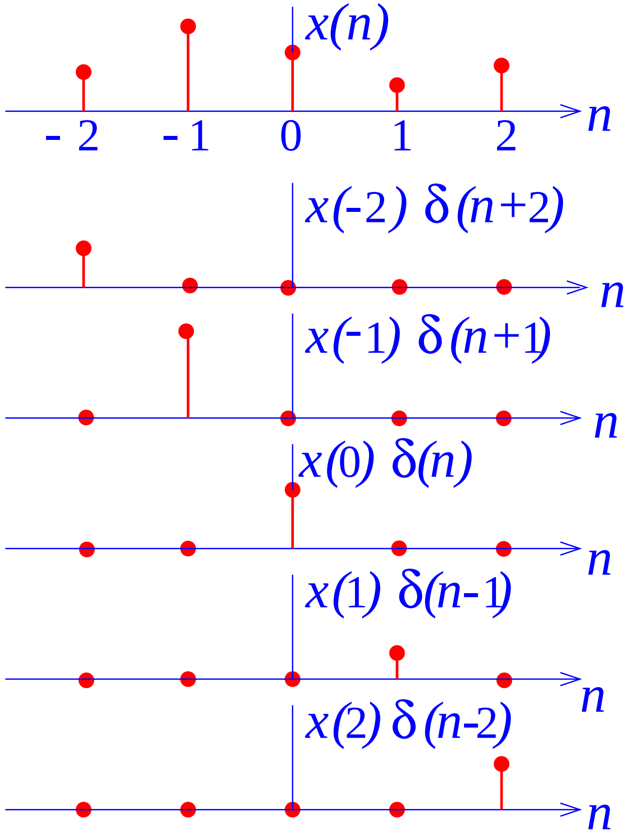
Hence, this system is nonlinear!

A time-invariant system has properties unvarying with time, i.e.:

$$\text{if} \quad y(n) = T\{x(n)\} \quad \Longrightarrow \quad y(n-k) = T\{x(n-k)\}$$

Linear Time-Invariant (LTI) system is a system that is both linear and time-invariant (sometimes referred to as a Linear Shift-Invariant (LSI) system)

2.2 Digital Signals via Impulse Functions



$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

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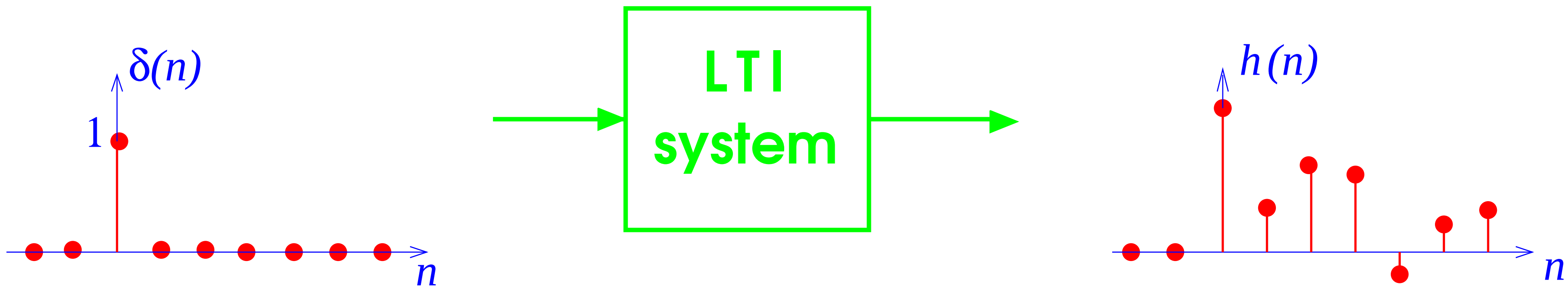
Let $h(n)$ be the response of the system to $\delta(n)$.

Due to the time-invariance property, the response to $\delta(n-k)$ is simply $h(n-k) \quad \Longrightarrow$

$$
\begin{aligned}
y(n) &= T\left\{x(n)\right\} \\
&= T\left\{\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right\} \\
&= \sum_{k=-\infty}^{\infty} x(k) T\left\{\delta(n-k)\right\} \\
&= \sum_{k=-\infty}^{\infty} x(k) h(n-k) = \{x(n)\} * \{h(n)\} \quad \text{convolution sum}
\end{aligned}
$$

The sequence $\{h(n)\}$ is commonly referred to as impulse response of the LTI system



An important property of convolution:

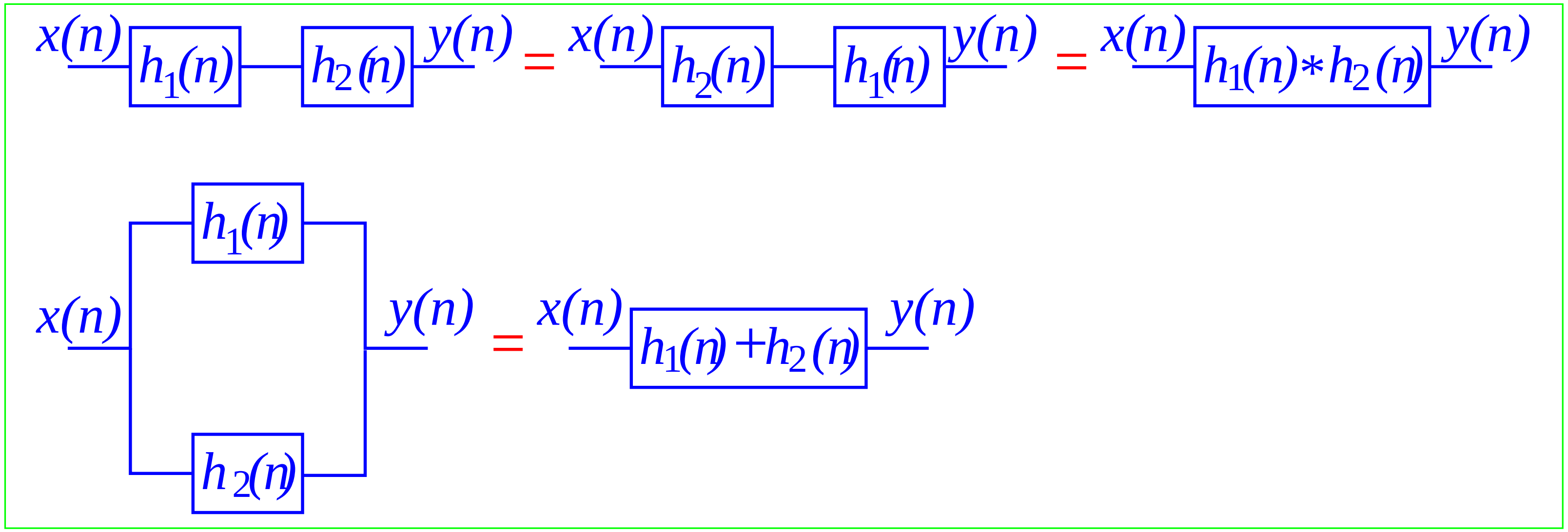
$$\{x(n)\} * \{h(n)\} = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$
$$= \{h(n)\} * \{x(n)\} \quad \Longrightarrow$$

the order in which two sequencies are convolved is unimportant!

Other properties of convolution:

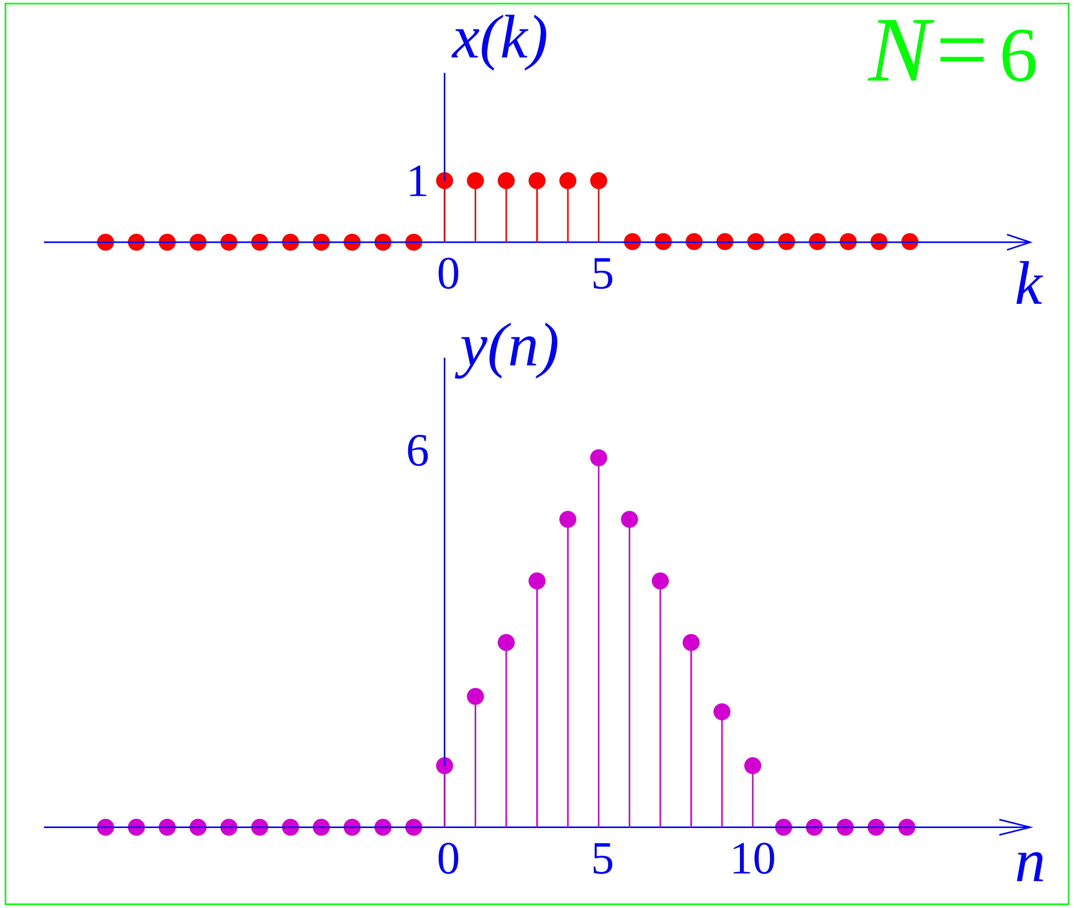
$$\begin{aligned} &\{x(n)\} * [\{h_1(n)\} * \{h_2(n)\}] \\ = \;& [\{x(n)\} * \{h_1(n)\}] * \{h_2(n)\} \end{aligned}$$ associativity

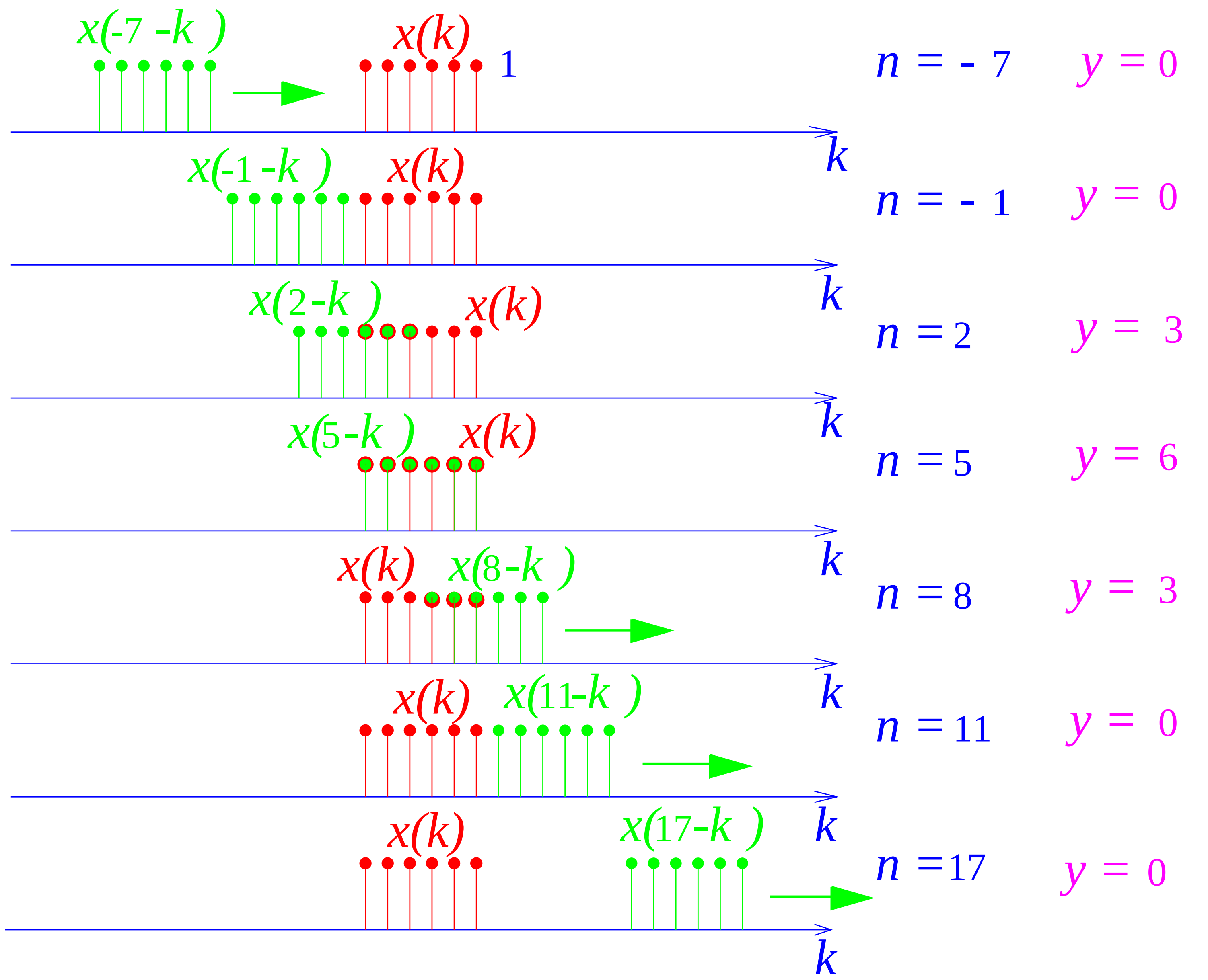
$$\begin{aligned} &\{x(n)\} * [\{h_1(n)\} + \{h_2(n)\}] \\ = \;& \{x(n)\} * \{h_1(n)\} + \{x(n)\} * \{h_2(n)\} \end{aligned}$$ distributivity



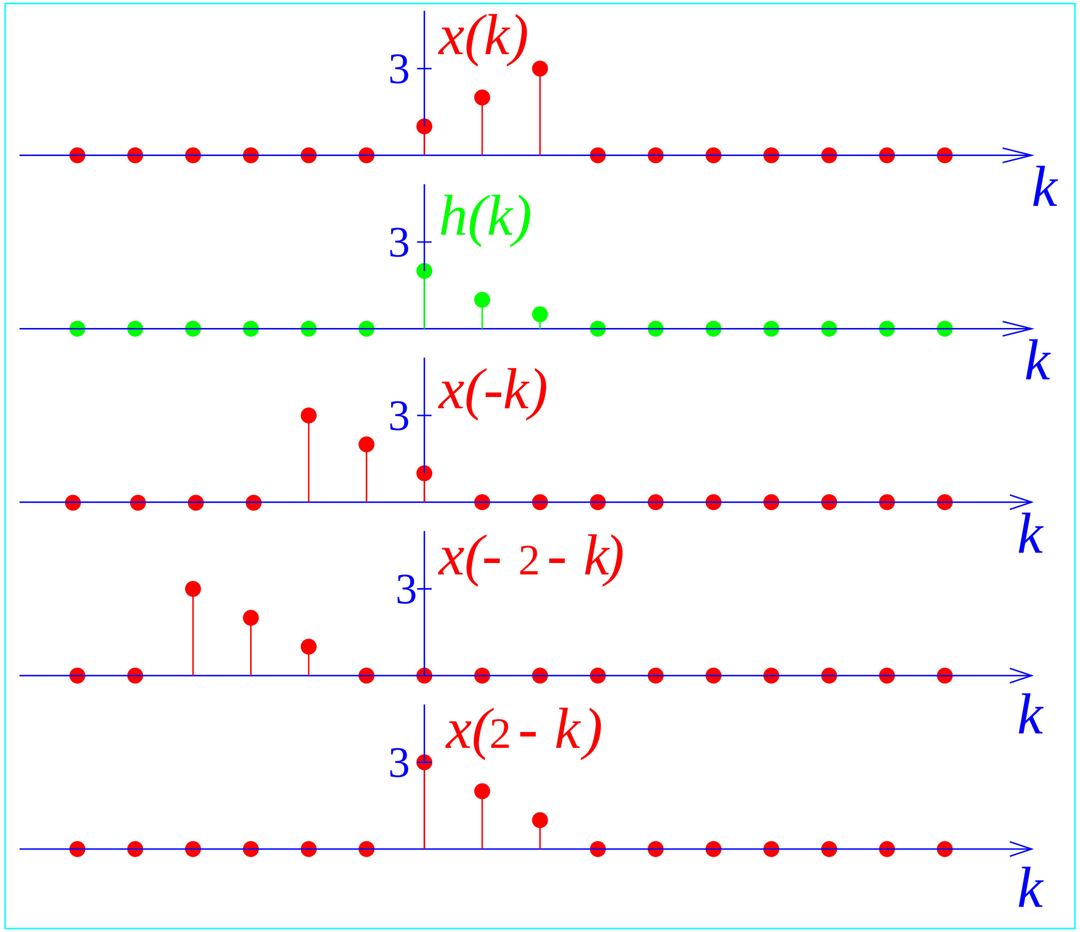
Example: Convolution of two rectangles:

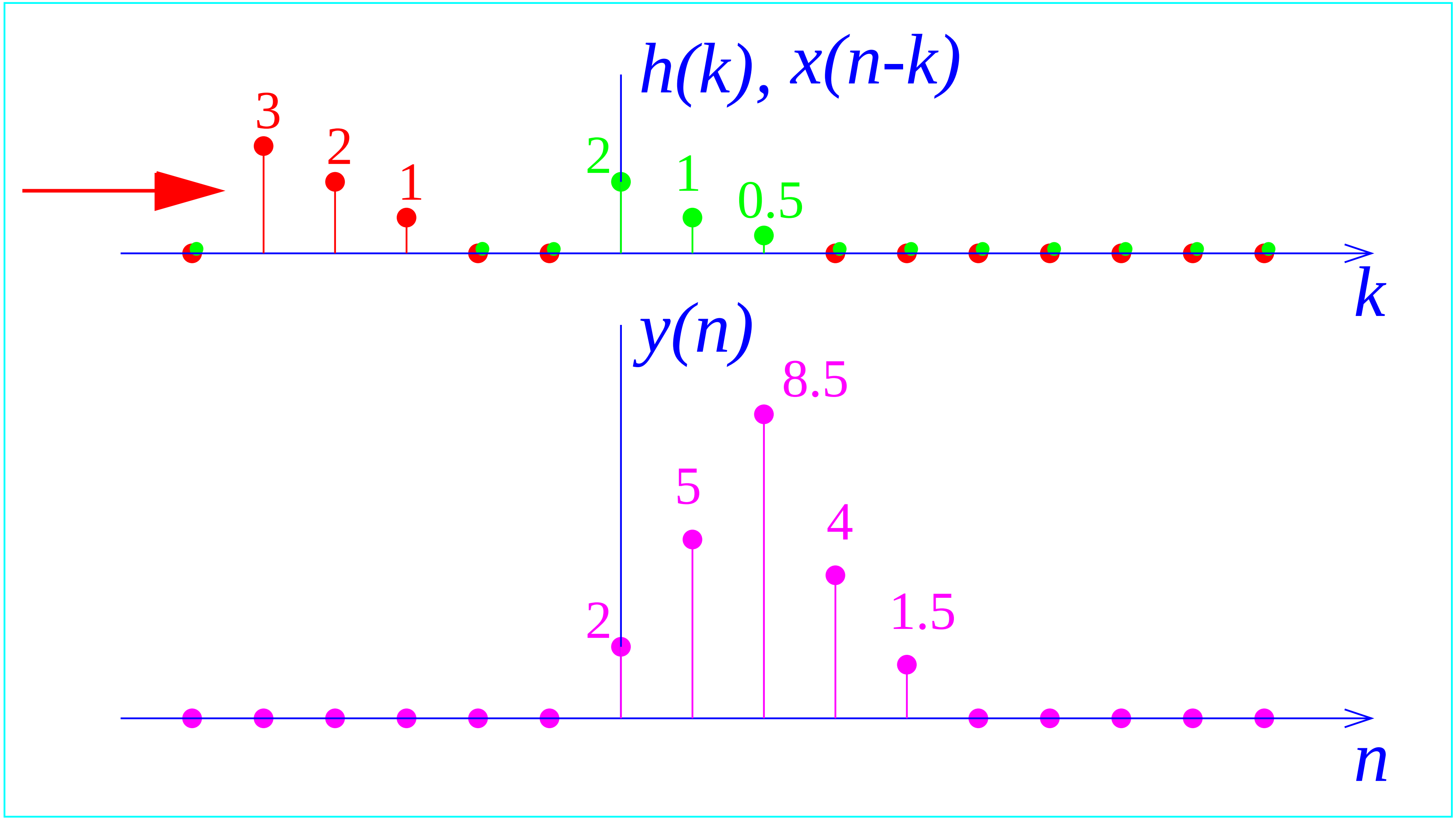
$$y(n) = \{x(n)\} * \{x(n)\}$$




x(-7 -k)
x(k)
1
n = - 7
y = 0
k
x(-1 -k)
x(k)
n = - 1
y = 0
k
x(2-k)
x(k)
n = 2
y = 3
k
x(5-k)
x(k)
n = 5
y = 6
k
x(k)
x(8-k)
n = 8
y = 3
k
x(k)
x(11-k)
n = 11
y = 0
k
x(k)
x(17-k)
n =17
y = 0
k

Example: Convolution of two sequencies $\{x(n)\} = \{\ldots, 0, 1, 2, 3, 0, \ldots\}$ and $\{h(n)\} = \{\ldots, 0, 2, 1, 0.5, 0, \ldots\}$




h(k), x(n-k)
3
2
1
2
1
0.5
k
y(n)
8.5
5
4
2
1.5
n

Result: LTI systems are stable if and only if

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

Proof of "if": Let the input $x(n)$ be bounded so that $|x(n)| < L_x$, $\forall n \in [-\infty, \infty]$. Then

$$\begin{aligned} |y(n)| &= \left| \sum_{k=-\infty}^{\infty} h(k)x(n-k) \right| \\ &\leq \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)| \\ &\leq L_x \sum_{k=-\infty}^{\infty} |h(k)| \implies |y(n)| < \infty \text{ if } \sum_{k=-\infty}^{\infty} |h(k)| < \infty \end{aligned}$$

Now, it remains to prove that if

$$\sum_{k=-\infty}^{\infty} |h(k)| = \infty$$

then a bounded input can be found that will cause an unbounded output.

Consider

$$x(n) = \begin{cases} h^*(-n)/|h(-n)|\,, & h(n) \neq 0 \\ 0\,, & h(n) = 0; \end{cases} \qquad \Longrightarrow$$

$$y(0) = \sum_{k=-\infty}^{\infty} x(-k)h(k) = \sum_{k=-\infty}^{\infty} |h(k)| \qquad \Longrightarrow$$

if $\sum_{k=-\infty}^{\infty} |h(k)| = \infty$, the output sequence is unbounded.

Definition: A causal system is one for which the output $y(n)$ depends on the inputs $\{\ldots, x(n-2), x(n-1), x(n)\}$ only.

Result: An LTI system is causal if and only if its impulse response $h(n) = 0$ for $n < 0$.

Proof of "if": From the definition of a causal system,

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ &= \sum_{k=0}^{\infty} h(k)x(n-k) \end{aligned}$$

Obviously, this equation is valid if $\sum_{k=-\infty}^{-1} h(k)x(n-k) = 0$ i.e., if $h(n) = 0$ for $n < 0$.

Now, it remains to prove that if $h(n) \neq 0$ for $n < 0$, than the system can be noncausal. Let

$$\begin{aligned} h(n) &= 0\,, \quad n < -1 \\ h(-1) &\neq 0 \qquad\qquad \Longrightarrow \end{aligned}$$

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) + h(-1)x(n+1) \quad \Longrightarrow$$

$y(n)$ depends on $x(n+1)$ $\Longrightarrow$ the system is noncausal.

Example: An LTI system with

$$h(n) = a^n u(n) = \begin{cases} a^n \,, & n \geq 0 \\ 0 \,, & n < 0; \end{cases}$$

- since $h(n) = 0$ for $n < 0$, the system is causal
- To decide on stability, we must compute the sum

$$S = \sum_{k=-\infty}^{\infty} |h(k)| = |a|^k = \begin{cases} \frac{1}{1-|a|} \,, & |a| < 1 \\ \infty \,, & |a| \geq 1; \end{cases} \qquad \Longrightarrow$$

the system is stable only for $|a| < 1$

2.3 Linear Constant-Coefficient Difference (LCCD) Equations

Consider LTI systems satisfying

$$\sum_{k=0}^{N} a(k)y(n-k) = \sum_{k=0}^{M} b(k)x(n-k) \qquad \text{ARMA}$$

Particular cases:

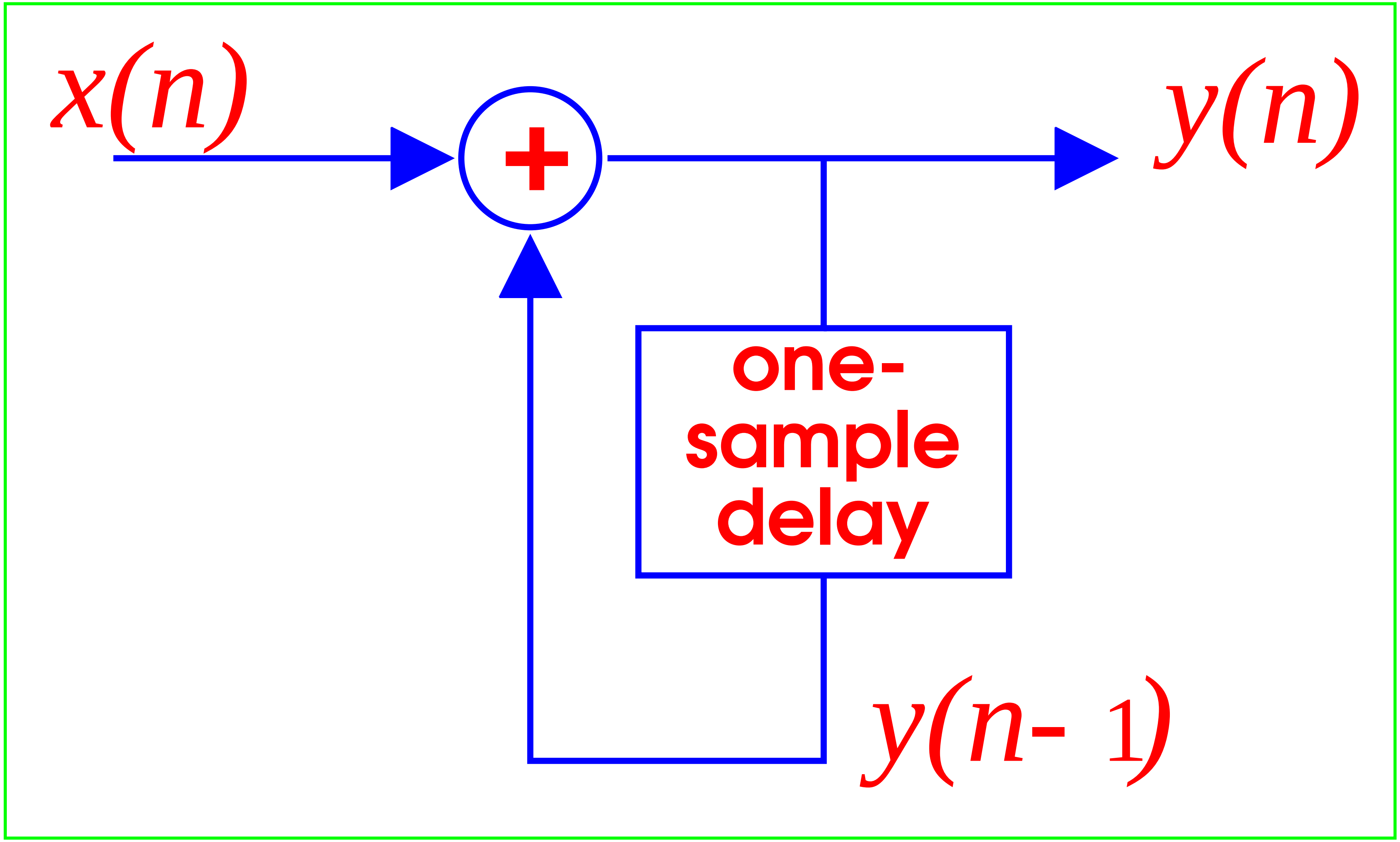
$$y(n) = \sum_{k=0}^{M} b(k)x(n-k) \qquad \text{MA}$$

$$\sum_{k=0}^{N} a(k)y(n-k) = x(n) \qquad \text{AR}$$

Example:

$$y(n) = \sum_{k=-\infty}^{n} x(k) \qquad \text{accumulator}$$

$$\begin{aligned}
y(n) - y(n-1) &= \sum_{k=-\infty}^{n} x(k) - \sum_{k=-\infty}^{n-1} x(k) \\
&= x(n) + \left\{ \sum_{k=-\infty}^{n-1} x(k) - \sum_{k=-\infty}^{n-1} x(k) \right\} \\
&= x(n)
\end{aligned}$$


x(n)
+
y(n)
one-
sample
delay
y(n- 1)

Property: MA systems are bounded-input bounded-output (BIBO) stable, i.e.

$$|y(n)| = \left| \sum_{k=0}^{M} b(k)x(n-k) \right| \leq \sum_{k=0}^{M} |b(k)|\,|x(n-k)| < \infty$$

for any bounded input $|x(n)| < \infty$ and coefficient sequence $|b(n)| < \infty$.

Remark: AR systems may be unstable. For example, the system

$$y(n) = ay(n-1) + x(n)$$

is unstable for $a > 1$, because $y(n)$ is unbounded for bounded $x(n)$.

Property: MA systems have finite impulse response (FIR), whereas AR systems have infinite impulse response (IIR).

Proof for MA systems:

$$h_{\mathrm{MA}}(n) = \begin{cases} 0\,, & n < 0 \\ b(n)\,, & 0 \le n \le M; \\ 0\,, & n > M \end{cases} \quad \Longrightarrow \quad \text{FIR!}$$

Proof for AR systems: $y(n)$ depends on $y(n-k)$, $k = 1, \ldots, \infty \quad \Longrightarrow$ $y(n)$ depends on $x(n-k)$, $k = 0, \ldots, \infty \quad \Longrightarrow$ the impulse response $h_{\mathrm{AR}}(n)$ is infinite, i.e., is in general case nonzero for all $n > 0$.

Result: Suppose that for a given input $x_p(n)$ we have found one particular output sequence $y_{\mathrm{P}}(n)$ so that a LCCD equation is satisfied. Then, the same equation with the same input is satisfied by any output of the form

$$y(n) = y_{\mathrm{P}}(n) + y_{\mathrm{H}}(n)$$

where $y_{\mathrm{H}}(n)$ is any solution to the LCCD equation with zero input $x(n) = 0$.

Remark: $y_{\mathrm{P}}(n)$ and $y_{\mathrm{H}}(n)$ are referred to as the particular and homogeneous solutions, respectively.

Proof of Result: Taking the sum of

$$\sum_{k=0}^{N} a(k) y_{\mathrm{P}}(n-k) = \sum_{k=0}^{M} b(k) x(n-k)$$
$$\sum_{k=0}^{N} a(k) y_{\mathrm{H}}(n-k) = 0$$

we obtain

$$\sum_{k=0}^{N} a(k) y(n-k) = \sum_{k=0}^{M} b(k) x(n-k)$$

with $y(n) = y_{\mathrm{P}}(n) + y_{\mathrm{H}}(n)$. Result is proven.

Property: A LCCD equation does not provide a unique specification of the output for a given input.

Corollary: Auxiliary information or conditions are required to specify uniquely the output for a given input.

Example: Let auxiliary information be in the form of N sequential output values. Then,

- later values can be obtained by rearranging LCCD equation as a recursive relation running forward in n
- prior values can be obtained by rearranging LCCD equation as a recursive relation running backward in n.

LCCD equations as recursive procedures:

$$y(n) = \sum_{k=0}^{M} \frac{b(k)}{a(0)} x(n-k) - \sum_{k=1}^{N} \frac{a(k)}{a(0)} y(n-k) \qquad \text{forwards}$$

$$y(n-N) = \sum_{k=0}^{M} \frac{b(k)}{a(N)} x(n-k) - \sum_{k=0}^{N-1} \frac{a(k)}{a(N)} y(n-k) \quad \text{backwards}$$

Example: First-order AR system $y(n) = ay(n-1) + x(n)$ with the given input $x(n) = b\delta(n-1)$ and the auxiliary condition $y(0) = y_0$.

Forwards recursion:

$$\begin{aligned}
y(1) &= ay_0 + b\,, \\
y(2) &= ay(1) + 0 \\
&= a(ay_0 + b) = a^2 y_0 + ab\,, \\
y(3) &= a(a^2 y_0 + ab) = a^3 y_0 + a^2 b\,, \\
\cdots &\cdots \cdots\cdots\cdots\cdots\,, \\
y(n) &= a^n y_0 + a^{n-1} b
\end{aligned}$$

Remark that:

$$y(n-1) = a^{-1}\,(y(n) - x(n)) \qquad \Longrightarrow$$

Backwards recursion:

$$\begin{aligned} y(-1) &= a^{-1}(y_0 - 0) \\ &= a^{-1} y_0 \,, \\ y(-2) &= a^{-2} y_0 \,, \\ y(-3) &= a^{-3} y_0 \,, \\ \cdots\cdots & \cdots \cdots\cdots\cdots\,, \\ y(-n) &= a^{-n} y_0 \end{aligned}$$

Question: is this system LTI and causal?

Auxiliary result: A linear system requires that the output be zero for all time when the input is zero for all time.

Proof: Represent zero input as a product of zero constant $c = 0$ and (arbitrary) non-zero signal $x(n)$:

$$c\,x(n) = 0 \cdot x(n) = 0$$

Hence, for a linear system

$$y(n) = T\{c\,x(n)\} = c\,T\{x(n)\} = 0 \cdot T\{x(n)\} = 0$$

Result is proven.

From the backwards recursion it follows that

$$y(-n) = a^{-n} y_0 \neq 0 \quad \text{for} \quad a \neq 0\,, \;\; y_0 \neq 0$$

whereas $x(-n) = 0\,,\;\; n \geq 0 \quad \Longrightarrow$

according to Result, the system is nonlinear!

The system was implemented in both positive and negative directions beginning with $n = 0 \quad \Longrightarrow \quad$ the system is noncausal!

The forwards-backwards recursion can be rewritten for arbitrary n as

$$y(n) = a^n y_0 + a^{n-1} b u(n-1)$$

Hence, the shift of the input by n_0 samples,
$\tilde{x}(n) = x(n - n_0) = b\,\delta(n - n_0 - 1)$, gives

$$\tilde{y}(n) = a^n y_0 + a^{n-n_0-1} b u(n - n_0 - 1) \neq y(n - n_0) \qquad \Longrightarrow$$

the system is not time invariant!

Example: First-order AR system $y(n) = ay(n-1) + x(n)$ with the given input $x(n) = b\delta(n-1)$ and the auxiliary condition $y(0) = 0$.

Recursion:

$$\begin{aligned}
\cdots\cdots \quad & \cdots \quad \cdots\,, \\
y(-1) &= 0\,, \\
y(0) &= 0\,, \\
y(1) &= a \cdot 0 + b \\
&= b\,, \\
y(2) &= ab\,, \\
\cdots\cdots \quad & \cdots \quad \cdots\cdots\,, \\
y(n) &= a^{n-1} b
\end{aligned}$$

This recursion can be rewritten as:

$$y(n) = a^{n-1} b\, u(n-1)\,, \quad \forall\ n$$

It is easy to prove that this system is a causal LTI system $\Longrightarrow$

Linearity, time-invariance, and causality depend on auxiliary conditions!

3 FREQUENCY-DOMAIN ANALYSIS

3.1 Review of Continuous-Time Fourier Transform

Consider a continuous complex signal

$$x(t) \in [-T/2, T/2]$$

Let us represent it using an arbitrary orthonormal basis $\varphi_n(t)$:

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \varphi_n(t) \qquad (*)$$

Orthonormality condition

$$\frac{1}{T}\int_{-T/2}^{T/2} \varphi_n(t)\varphi_k^*(t)\, dt = \delta(n-k)$$

Multiplying $(*)$ with $\varphi_k^*(t)$ and integrating over the interval, we obtain

$$\begin{aligned}
\frac{1}{T}\int_{-T/2}^{T/2} x(t)\varphi_k^*(t)\,dt &= \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{\infty} \alpha_n \varphi_n(t)\varphi_k^*(t)\,dt \\
&= \sum_n \alpha_n \Big(\frac{1}{T}\int_{-T/2}^{T/2} \varphi_n(t)\varphi_k^*(t)\,dt\Big) \\
&= \sum_{n=-\infty}^{\infty} \alpha_n \delta(n-k) = \alpha_k \qquad \Longrightarrow
\end{aligned}$$

the coefficients of expansion are given by:

$$\alpha_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\varphi_k^*(t)\,dt$$

Result: The functions $\varphi_n(t) = \exp\{j2\pi nt/T\}$ are orthonormal at the interval $[-T/2, T/2]$.

Proof:

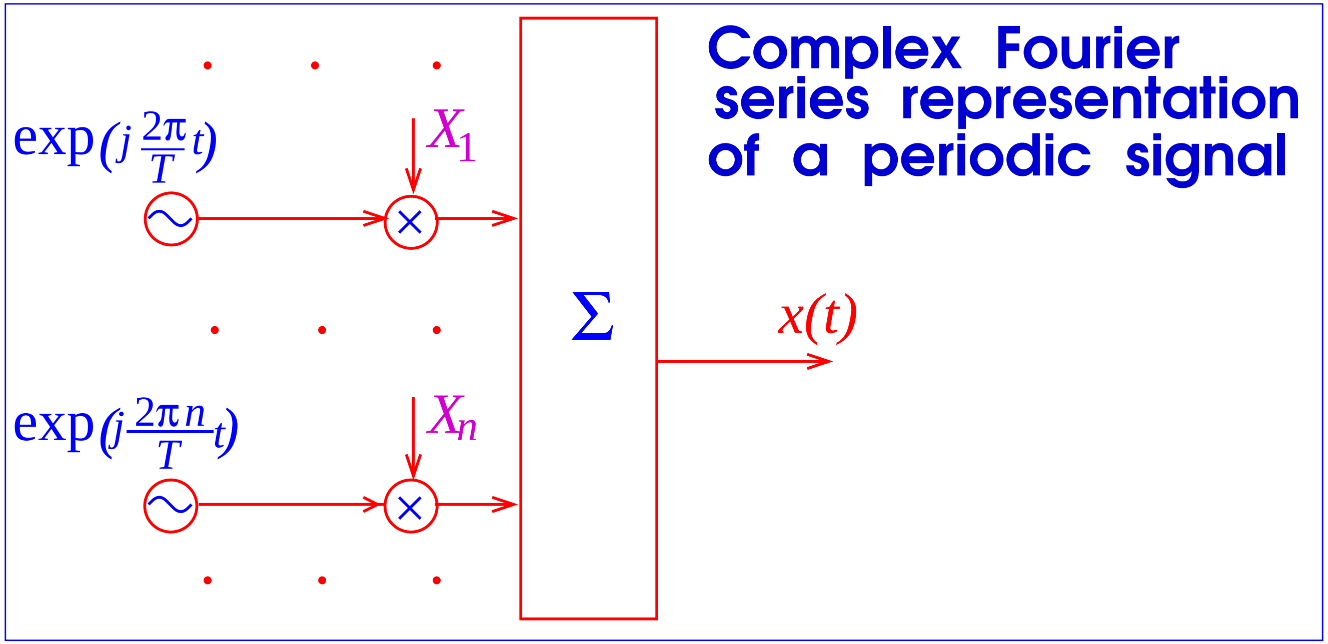
$$\frac{1}{T}\int_{-T/2}^{T/2} \varphi_n(t)\varphi_k^*(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{j\frac{2\pi(n-k)}{T}t}\,dt$$
$$= \frac{\sin(\pi(n-k))}{\pi(n-k)} = \delta(n-k)$$

Result is proven, i.e., we can take exponential functions $\varphi_n(t) = \exp\{j2\pi nt/T\}$ as orthonormal basis $\Longrightarrow$ we obtain Fourier Series!

Fourier Series for a periodic signal $x(t) = x(t+T)$:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \, e^{j\frac{2\pi n}{T}t}$$

$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi n}{T}t}\, dt$$



Fourier coefficients can be viewed as a signal spectrum:

$$X_n \sim X(\omega_n) \ , \quad \text{where} \quad \omega_n = \frac{2\pi n}{T} \qquad \Longrightarrow$$

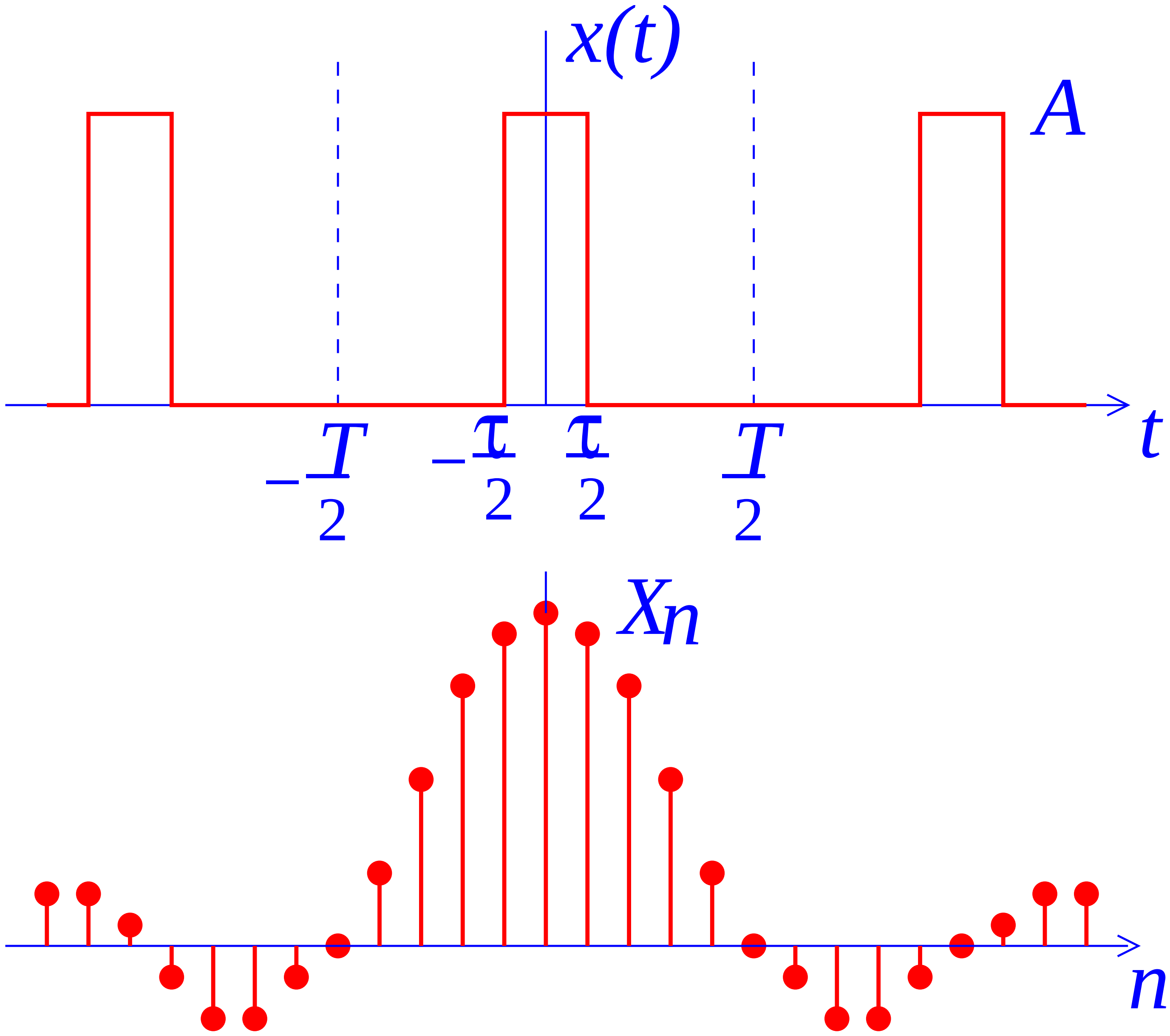
Fourier series can be applied for analysis of signal spectrum! Also, this interpretation shows that periodic signals have discrete spectrum.

Example: Periodic sequence of rectangles:

$$
\begin{aligned}
X_n &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi n}{T}t}\, dt \\
&= \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\, e^{-j\frac{2\pi n}{T}t}\, dt \\
&= \frac{A\tau}{T}\,\frac{\sin(\pi n\frac{\tau}{T})}{\pi n\frac{\tau}{T}} \qquad \text{real coefficients}
\end{aligned}
$$

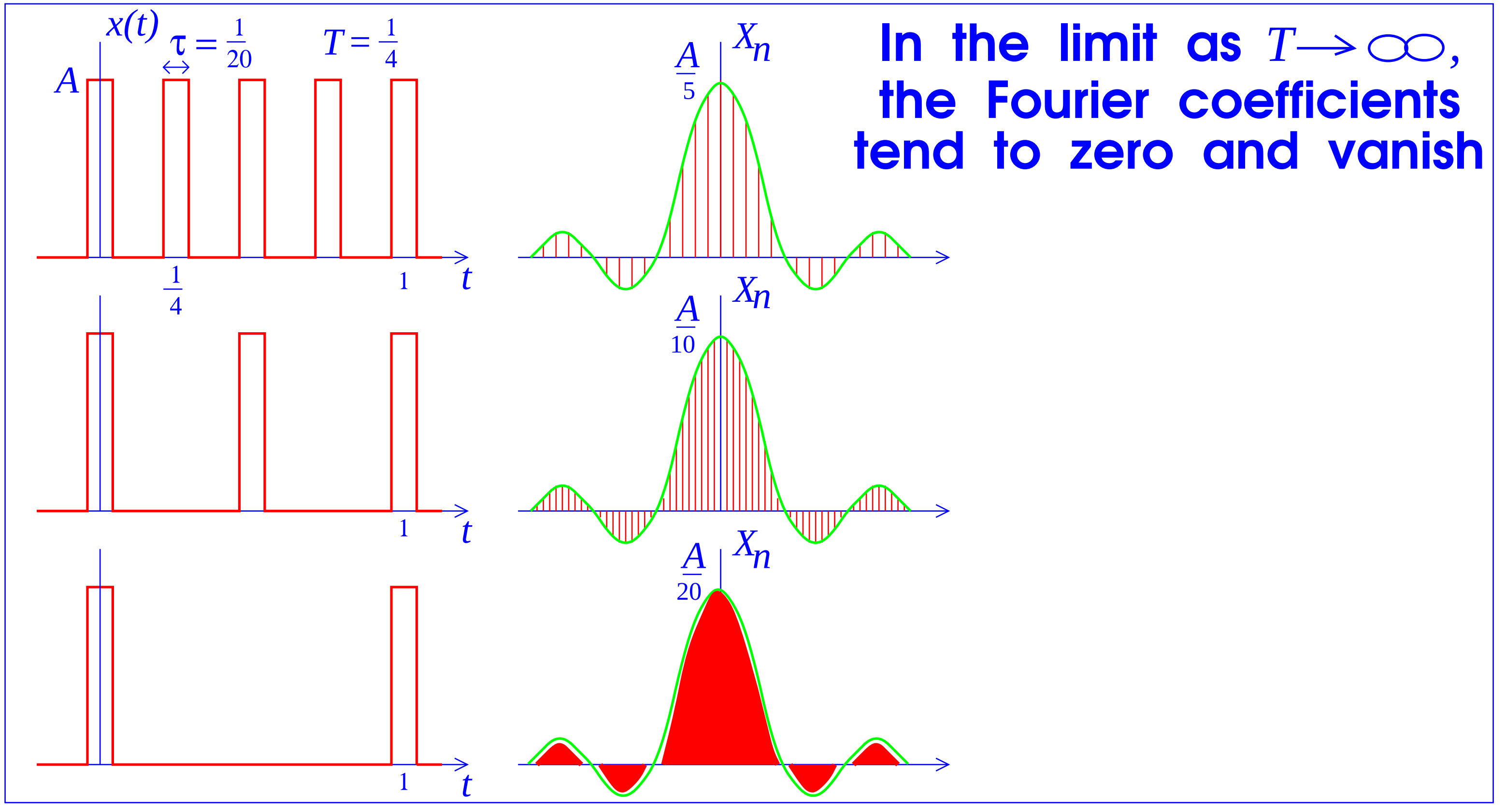
Remarks:

- in general Fourier coefficients are complex-valued
- for real signals $X_{-n} = X_n^*$
- alternative expressions for trigonometric Fourier series exist, exploiting summation of sine and cosine functions


$x(t)$
A
$-\frac{T}{2}$
$-\frac{\tau}{2}$
$\frac{\tau}{2}$
$\frac{T}{2}$
t
X_n
n

What about Fourier representations of nonperiodic continuous-time signals?

Assuming a finite-energy signal and $T \to \infty$ in the Fourier series, we obtain $\lim_{T\to\infty} X_n = 0$.



Trick: To preserve the Fourier coefficients from degradation at $T \rightarrow \infty$, introduce

$$\tilde{X}_n = T\, X_n = \int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi n}{T}t}\, dt$$

Transition to Fourier transform:

$$\begin{aligned} X(\omega) &= \lim_{T\rightarrow\infty} \tilde{X}_n \\ &= \lim_{T\rightarrow\infty} \int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi n}{T}t}\, dt \\ &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \end{aligned}$$

where the discrete frequency $\frac{2\pi n}{T}$ becomes the continuous frequency ω

Transition to inverse Fourier transform:

$$
\begin{aligned}
x(t) &= \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} X_n \, e^{j\frac{2\pi n}{T}t} \\
&= \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \frac{\tilde{X}_n}{T} \, e^{j\frac{2\pi n}{T}t} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega \qquad \leftarrow \quad d\omega = \frac{2\pi}{T}, \quad \omega = \frac{2\pi n}{T}
\end{aligned}
$$

Continuous-time Fourier transform (CTFT):

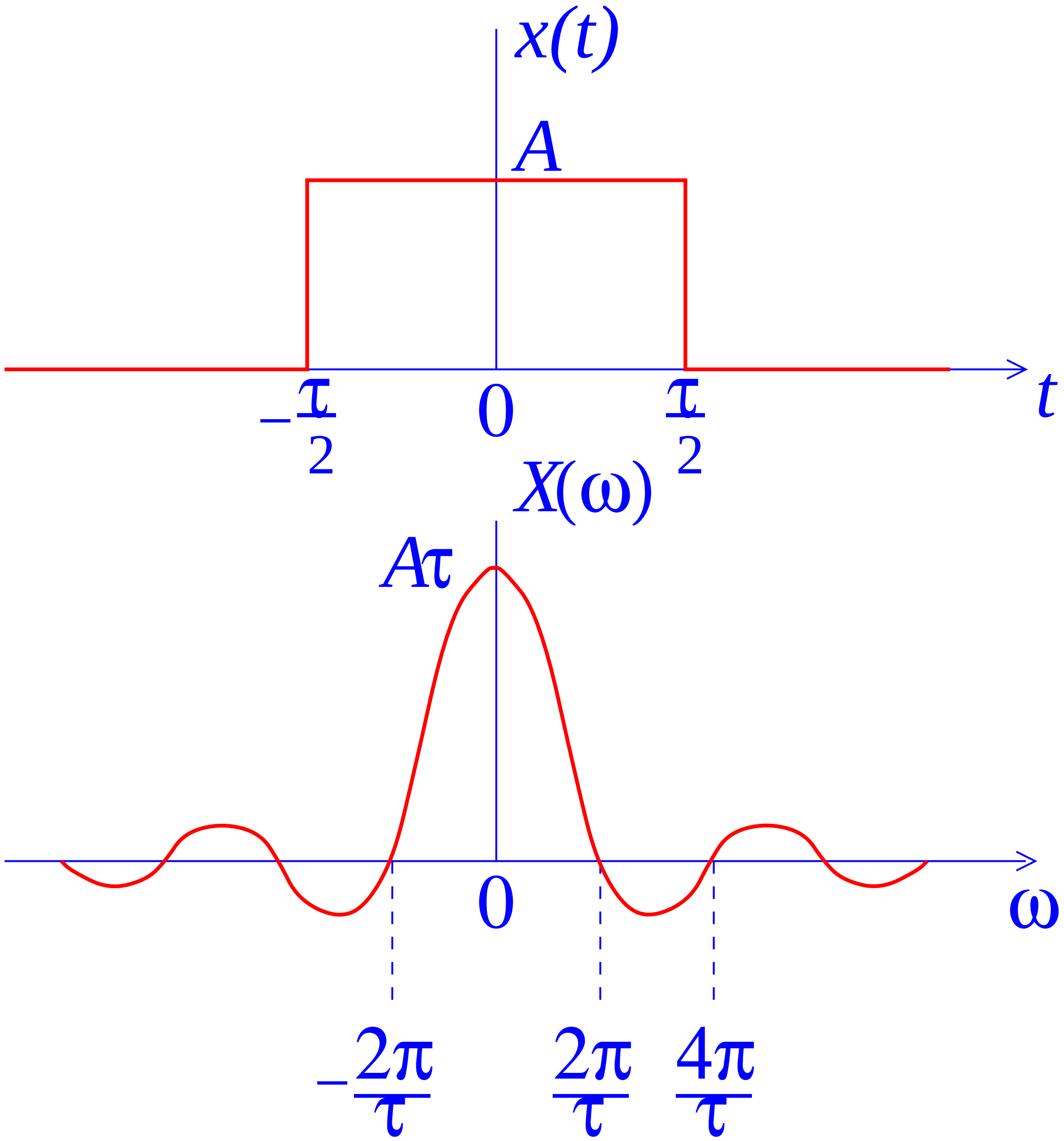
$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t) \, e^{-j\omega t} \, dt \\
x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega
\end{aligned}
$$

Example: Finite-energy rectangular signal:

$$
\begin{aligned}
X(\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\
&= \int_{-\tau/2}^{\tau/2} A\, e^{-j\omega t}\, dt \\
&= A\tau\, \frac{\sin(\omega\tau/2)}{\omega\tau/2} \qquad \text{real spectrum}
\end{aligned}
$$

Remarks:

- in general Fourier spectrum is complex-valued
- for real signals $X(\omega) = X^*(-\omega)$



$x(t)$
A
$-\frac{\tau}{2}$
0
$\frac{\tau}{2}$
t
$X(\omega)$
$A\tau$
0
ω
$-\frac{2\pi}{\tau}$
$\frac{2\pi}{\tau}$
$\frac{4\pi}{\tau}$

Definition of Dirac delta-function:

$$\delta(t) = \begin{cases} \infty\,, & t = 0 \\ 0\,, & t \neq 0 \end{cases}\,, \qquad \int\limits_{-\infty}^{\infty} \delta(t)\,dt = 1$$

Do not confuse continuous-time $\delta(t)$ and discrete time $\delta(n)$!

Shifting property:

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-\tau)\,dt = f(\tau)$$

Delta-function in time domain:

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} \, d\omega$$

The spectrum of $\delta(t - t_0)$:

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t) \, e^{-j\omega t} \, dt \\ &= \int_{-\infty}^{\infty} \delta(t - t_0) \, e^{-j\omega t} \, dt \\ &= e^{-j\omega t_0} \end{aligned}$$

Delta-function in frequency domain:

$$\delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\omega t}\, dt = \begin{cases} \infty\,, & \omega = 0 \\ 0\,, & \omega \neq 0 \end{cases}$$

Let the signal be

$$x(t) = A\, e^{j\omega_0 t}$$

Then,

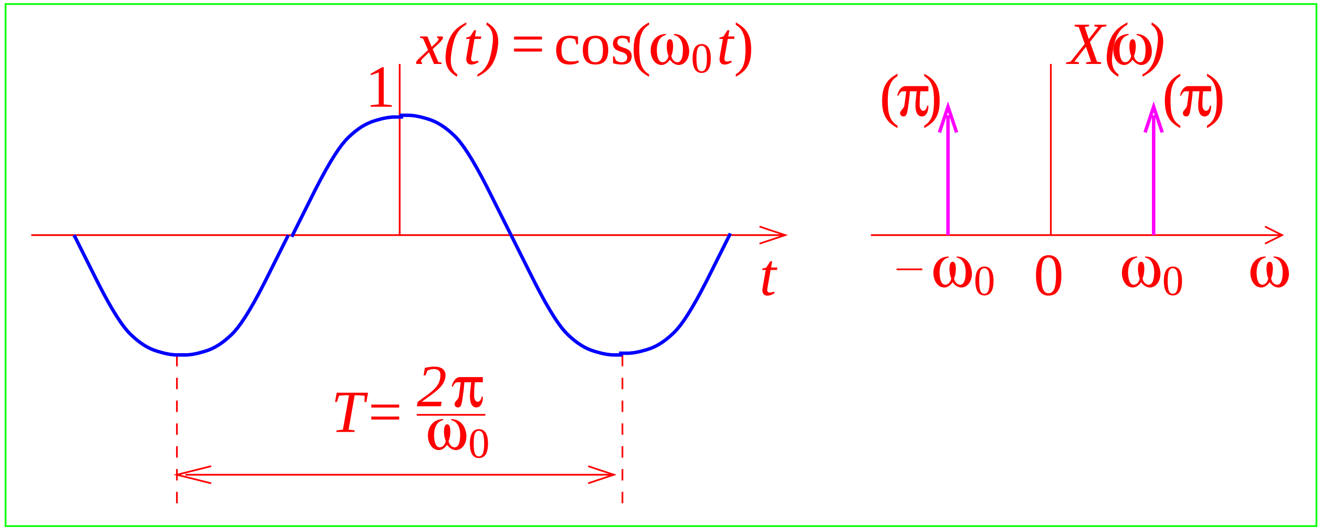
$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\ &= A \int_{-\infty}^{\infty} e^{-j(\omega-\omega_0)t}\, dt \\ &= A\, 2\pi\, \delta(\omega - \omega_0) \end{aligned}$$

Harmonic Fourier pairs:

$$e^{j\omega_0 t} \quad \leftrightarrow \quad 2\pi\,\delta(\omega - \omega_0)$$

$$\cos(\omega_0 t) \quad \leftrightarrow \quad \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$

$$\sin(\omega_0 t) \quad \leftrightarrow \quad \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$$



Parseval theorem for CTFT:

$$\int_{-\infty}^{\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega$$

Proof:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left\{\iint\limits_{-\infty}^{\infty} x(t)x^*(\tau)\,e^{-j\omega(t-\tau)}\,dt\,d\tau\right\}d\omega \\
&= \iint\limits_{-\infty}^{\infty} x(t)x^*(\tau)\underbrace{\left\{\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{-j\omega(t-\tau)}\,d\omega\right\}}_{\delta(t-\tau)}dt\,d\tau \\
&= \int_{-\infty}^{\infty} |x(t)|^2\,dt
\end{aligned}$$

Example: Parseval theorem for the harmonic Fourier pair $e^{j\omega_0 t} \leftrightarrow 2\pi\,\delta(\omega - \omega_0)$. The first part of Parseval equality:

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |e^{j\omega_0 t}|^2\, dt = \infty$$

Redefine delta-function as the following limit:

$$\delta(\omega) = \lim_{\Omega \to 0} p(\omega)$$

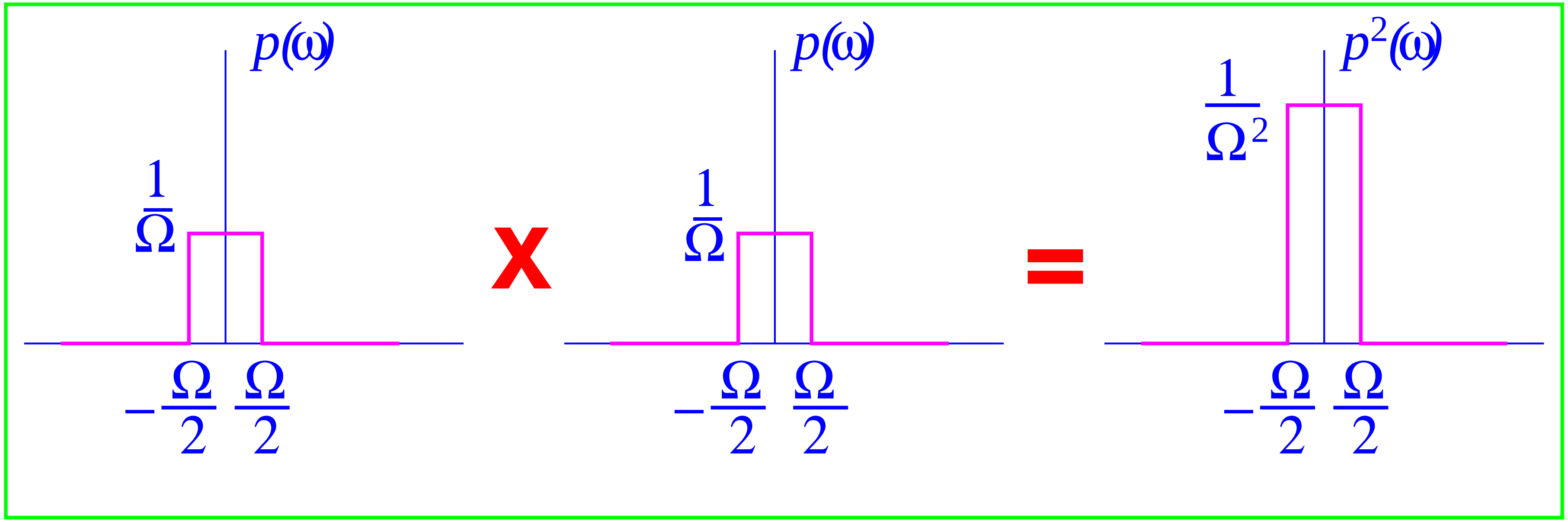
where the pulselike function

$$p(\omega) = \begin{cases} \frac{1}{\Omega}\,, & -\frac{\Omega}{2} \le \omega \le \frac{\Omega}{2} \\ 0\,, & \text{otherwise} \end{cases}$$

The second part of Parseval equality

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega &= 2\pi \int_{-\infty}^{\infty} \delta^2(\omega-\omega_0)\, d\omega \\
&= 2\pi \int_{-\infty}^{\infty} \lim_{\Omega\to 0} p^2(\omega-\omega_0)\, d\omega \\
&= 2\pi \lim_{\Omega\to 0} \int\limits_{\omega_0-\frac{\Omega}{2}}^{\omega_0+\frac{\Omega}{2}} \frac{1}{\Omega^2}\, d\omega \\
&= 2\pi \lim_{\Omega\to 0} \frac{1}{\Omega} = \infty
\end{aligned}
$$

Multiplication of two identical pulselike functions



3.2 Discrete-Time Fourier Transform

Represent continuous signal $x(t)$ via discrete sequence $x(n)$:

$$x(t) = \sum_{n=-\infty}^{\infty} x(n\,\Delta t)\delta(t - n\,\Delta t)$$
$$= \sum_{n=-\infty}^{\infty} x(n)\delta(t - n\,\Delta t)$$

Substituting this equation in CTFT, we obtain:

$$X(\omega) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(n)\delta(t - n\,\Delta t)e^{-j\omega t}\,dt$$
$$= \sum_{n=-\infty}^{\infty} x(n) \int_{-\infty}^{\infty} \delta(t - n\,\Delta t)e^{-j\omega t}\,dt = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n\,\Delta t}$$

In DTFT, let us use the normed frequency i.e., let $\omega \longrightarrow \omega\,\Delta t$:

$$\omega_{\mathrm{DTFT}} = \omega_{\mathrm{CTFT}}\,\Delta t$$

Hence, the last expression for $X(\omega)$ can be rewritten as

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

$X(\omega)$ is periodic with 2π:

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\underbrace{e^{-j2\pi n}}_{1} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi)n} = X(\omega+2\pi) \end{aligned}$$

Trick: use in DTFT only one period of $X(\omega)$:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad \text{DTFT}$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n}\, d\omega \qquad \text{Inverse DTFT}$$

$$\begin{aligned} x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n}\, d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{m=-\infty}^{\infty} x(m)\, e^{j\omega(n-m)}\, d\omega \\ &= \sum_{m=-\infty}^{\infty} x(m) \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-m)}\, d\omega}_{\delta(n-m)} = x(n) \end{aligned}$$

Compare Fourier series and DTFT:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \, e^{j\frac{2\pi n}{T}t}, \qquad X_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-j\frac{2\pi n}{T}t}\, dt \quad \text{FS}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}, \qquad x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n}\, d\omega \quad \text{DTFT}$$

Observation: replacing in Fourier series

$$x(t) \longrightarrow X(\omega) \qquad X_n \longrightarrow x(n) \qquad t \longrightarrow -\omega \qquad T \longrightarrow 2\pi$$

we obtain DTFT!!!

An important conclusion follows: DTFT is equivalent to Fourier series but applied to the "opposite" domain. In Fourier series, a periodic continuous signal is represented as a sum of exponentials weighted by discrete Fourier (spectral) coefficients. In DTFT, a periodic continuous spectrum is represented as a sum of exponentials weighted by discrete signal values.

Remarks:

- DTFT can be derived directly from the Fourier series
- all developments for Fourier series can be applied to DTFT
- the relationship between Fourier series and DTFT illustrates the duality between time and frequency domains

Parseval theorem for DTFT:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$$

Proof:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x(n)x^*(m)e^{-j\omega(n-m)}\, d\omega \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x(n)x^*(m)\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega(n-m)}\, d\omega}_{\delta(n-m)} \\
&= \sum_{n=-\infty}^{\infty} |x(n)|^2
\end{aligned}$$

When does the DTFT exist ($|X(\omega)| < \infty$)?

Sufficient condition:

$$\sum_{n=-\infty}^{\infty} |x(n)| \leq \infty$$

Proof:

$$\begin{aligned} |X(\omega)| &= \left| \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right| \\ &\leq \sum_{n=-\infty}^{\infty} |x(n)| \underbrace{|e^{-j\omega n}|}_{1} \\ &= \sum_{n=-\infty}^{\infty} |x(n)| \leq \infty \end{aligned}$$

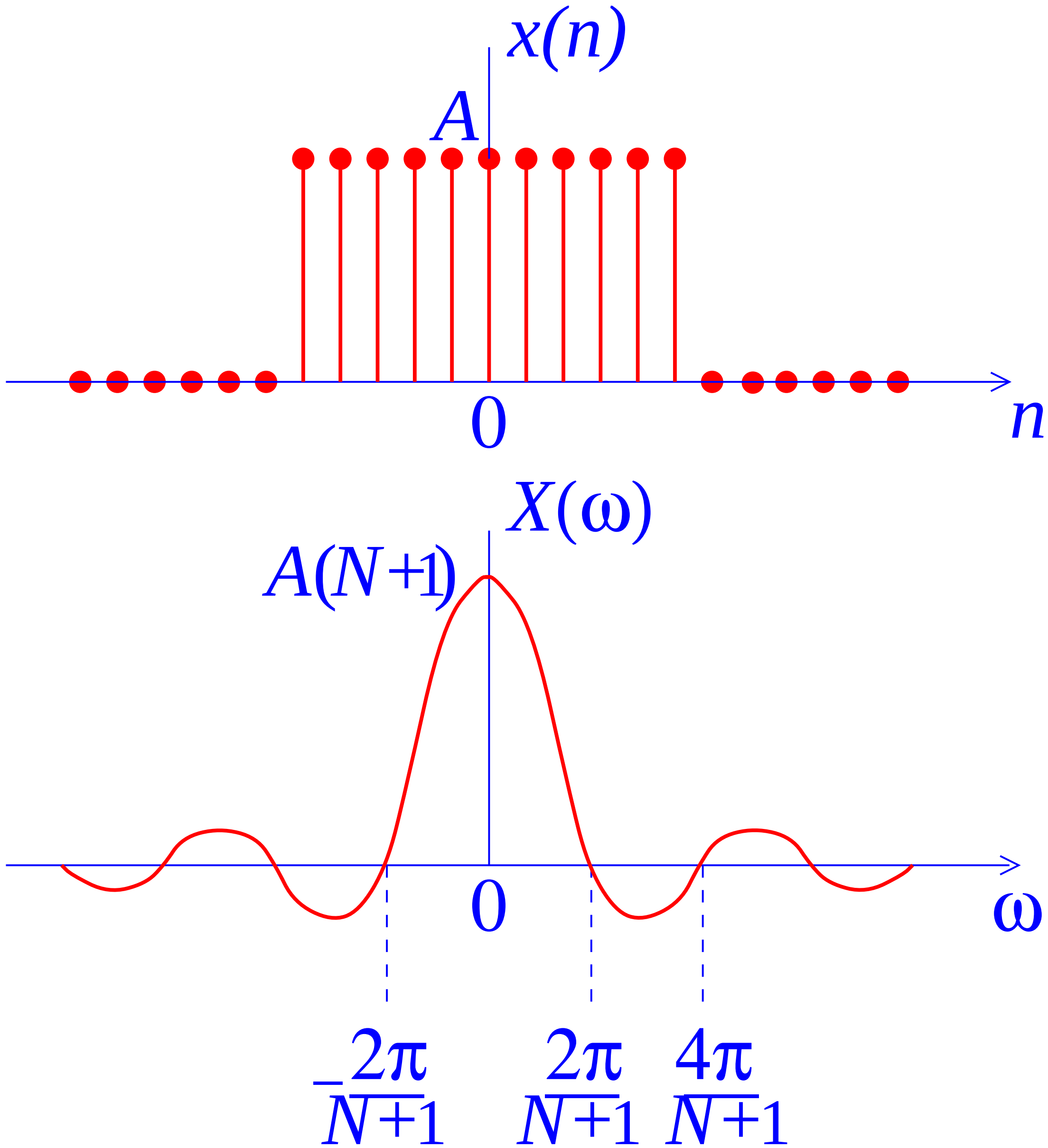
Example: Finite-energy rectangular signal:

$$X(\omega) = \sum_{n=-N/2}^{N/2} Ae^{-j\omega n} = A \sum_{n=-N/2}^{N/2} e^{-j\omega n}$$

$$= A\,(N+1)\,\frac{\sin\left(\frac{N+1}{2}\,\omega\right)}{(N+1)\sin\left(\frac{\omega}{2}\right)}$$

$$\simeq A\,(N+1)\,\underbrace{\frac{\sin\left(\frac{N+1}{2}\,\omega\right)}{\frac{N+1}{2}\,\omega}}_{\text{well--known function}} \qquad \text{for } \omega \ll \pi$$

Both functions look very similar in their "mainlobe" domain!


x(n)
A
0
n
X(ω)
A(N+1)
0
ω
$-\frac{2\pi}{N+1}$
$\frac{2\pi}{N+1}$
$\frac{4\pi}{N+1}$

3.3 Properties of DTFT

Linearity:

$$\text{If} \quad X(\omega) = \mathcal{F}\{x(n)\} \quad \text{and} \quad Y(\omega) = \mathcal{F}\{y(n)\}$$

$$\text{then} \quad a\,X(\omega) + b\,Y(\omega) = a\,\mathcal{F}\{x(n)\} + b\,\mathcal{F}\{y(n)\}$$

$$\text{Also if} \quad x(n) = \mathcal{F}^{-1}\{X(\omega)\} \quad \text{and} \quad y(n) = \mathcal{F}^{-1}\{Y(\omega)\}$$

$$\text{then} \quad a\,x(n) + b\,y(n) = a\,\mathcal{F}^{-1}\{X(\omega)\} + b\,\mathcal{F}^{-1}\{Y(\omega)\}$$

Proof: elementary (direct substitution).

Time shifting:

$$\text{If} \quad X(\omega) = \mathcal{F}\{x(n)\} \quad \text{then} \quad X(\omega)e^{-j\omega m} = \mathcal{F}\{x(n-m)\}$$

$$\text{Also if} \quad x(n) = \mathcal{F}^{-1}\{X(\omega)\} \text{ then } x(n-m) = \mathcal{F}^{-1}\{X(\omega)e^{-j\omega m}\}$$

Proof:

$$\mathcal{F}\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(\underbrace{n-m}_{k})\, e^{-j\omega n} = \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega(m+k)}$$

$$= e^{-j\omega m} \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k} = X(\omega)e^{-j\omega m}$$

$$\mathcal{F}^{-1}\{X(\omega)e^{-j\omega m}\} = \mathcal{F}^{-1}\{\mathcal{F}\{x(n-m)\}\} = x(n-m)$$

Frequency shifting:

$$\text{If} \quad X(\omega) = \mathcal{F}\{x(n)\} \quad \text{then} \quad X(\omega - \nu) = \mathcal{F}\{x(n)e^{j\nu n}\}$$

$$\text{Also if} \quad x(n) = \mathcal{F}^{-1}\{X(\omega)\} \ \text{then} \ x(n)e^{j\nu n} = \mathcal{F}^{-1}\{X(\omega - \nu)\}$$

Proof:

$$\mathcal{F}\{x(n)e^{j\nu n}\} = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega-\nu)n} = X(\omega - \nu)$$

$$\mathcal{F}^{-1}\{X(\omega - \nu)\} = \mathcal{F}^{-1}\{\mathcal{F}\{\, x(n)e^{j\nu n}\}\} = x(n)e^{j\nu n}$$

Time reversal:

$$\text{If} \quad X(\omega) = \mathcal{F}\{x(n)\} \quad \text{then} \quad X(-\omega) = \mathcal{F}\{x(-n)\}$$

$$\text{Also if} \quad x(n) = \mathcal{F}^{-1}\{X(\omega)\} \ \text{then} \ x(-n) = \mathcal{F}^{-1}\{X(-\omega)\}$$

Proof:

$$\mathcal{F}\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(\underbrace{-n}_{m})e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x(m)e^{j\omega m} = X(-\omega)$$

$$\mathcal{F}^{-1}\{X(-\omega)\} = \mathcal{F}^{-1}\{\mathcal{F}\{\, x(-n)\}\} = x(-n)$$

Differentiation in frequency:

$$\text{If} \quad X(\omega) = \mathcal{F}\{x(n)\} \quad \text{then} \quad j\,\frac{dX(\omega)}{d\omega} = \mathcal{F}\{n\,x(n)\}$$

$$\text{Also if} \quad x(n) = \mathcal{F}^{-1}\{X(\omega)\} \;\; \text{then} \;\; n\,x(n) = \mathcal{F}^{-1}\{j\,\frac{dX(\omega)}{d\omega}\}$$

Proof:

$$\begin{aligned}
\mathcal{F}\{n\,x(n)\} &= \sum_{n=-\infty}^{\infty} nx(n)e^{-j\omega n} = j\sum_{n=-\infty}^{\infty} x(n)\,\frac{d(e^{-j\omega n})}{d\omega} \\
&= j\frac{d}{d\omega}\Big\{\sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n}\Big\} = j\,\frac{dX(\omega)}{d\omega} \\
\mathcal{F}^{-1}\{j\,\frac{dX(\omega)}{d\omega}\} &= \mathcal{F}^{-1}\{\mathcal{F}\{\,n\,x(n)\}\} = n\,x(n)
\end{aligned}$$

Convolution theorem:

$$\text{If} \quad X(\omega) = \mathcal{F}\{x(n)\} \quad , \quad H(\omega) = \mathcal{F}\{h(n)\} \quad ,$$

$$\text{and} \quad y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \{x(n)\} * \{h(n)\}$$

$$\text{then} \quad Y(\omega) = \mathcal{F}\{y(n)\} = X(\omega)H(\omega)$$

Convolution of sequencies in time-domain is equivalent to multiplication of the corresponding Fourier transforms in frequency domain.

Proof of convolution theorem:

$$
\begin{aligned}
Y(\omega) &= \mathcal{F}\{y(n)\} = \sum_{n=-\infty}^{\infty}\left\{\sum_{k=-\infty}^{\infty} x(k)h(\underbrace{n-k}_{m})\right\} e^{-j\omega n} \\
&= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x(k)h(m)e^{-j\omega(m+k)} \\
&= \left\{\sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k}\right\}\left\{\sum_{m=-\infty}^{\infty} h(m)e^{-j\omega m}\right\} \\
&= X(\omega)H(\omega)
\end{aligned}
$$

Windowing theorem:

$$\text{If} \quad X(\omega) = \mathcal{F}\{x(n)\} \quad , \quad W(\omega) = \mathcal{F}\{w(n)\} \quad ,$$

$$\text{and} \quad y(n) = x(n)w(n)$$

$$\text{then} \quad Y(\omega) = \mathcal{F}\{y(n)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\nu)W(\omega-\nu)\,d\nu$$

Multiplication of sequencies in time-domain is equivalent to periodic convolution of the corresponding Fourier transforms in frequency domain.

Proof: by means of direct substitution, similarly to the proof of convolution theorem.

Generalised Parseval theorem:

$$\text{If} \quad X(\omega) = \mathcal{F}\{x(n)\} \quad , \quad Y(\omega) = \mathcal{F}\{y(n)\} \quad ,$$

$$\text{then} \quad \sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)Y^*(\omega)\, d\omega$$

Proof: similarly to the proof of Parseval theorem.

Summary of main properties of DTFT

Sequence $x(n)$	Fourier Transform $X(\omega)$
$a\,x(n) + b\,y(n)$	$a\,X(\omega) + b\,Y(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x(n-m)$	$e^{-j\omega m}X(\omega)$
$e^{j\nu n}x(n)$	$X(\omega-\nu)$
$x(-n)$	$X(-\omega)$
$n\,x(n)$	$j\,\frac{dX(\omega)}{d\omega}$
$\{x(n)\} * \{h(n)\}$	$X(\omega)H(\omega)$
$x(n)w(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\nu)W(\omega-\nu)\,d\nu$
$\sum_{n=-\infty}^{\infty} \|x(n)\|^2$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} \|X(\omega)\|^2\,d\omega$
$\sum_{n=-\infty}^{\infty} x(n)y^*(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)Y^*(\omega)\,d\omega$

3.4 Frequency-Domain Representation of Discrete-Time Signals and Systems

Recall impulse response $h(n)$ of an LTI system:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Consider an input sequence $x(n) = e^{j\omega n}, \quad -\infty < n < \infty$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega(n-k)} = e^{j\omega n}\underbrace{\left\{ \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \right\}}_{H(\omega)}$$

$$= e^{j\omega n}H(\omega)$$

The complex function

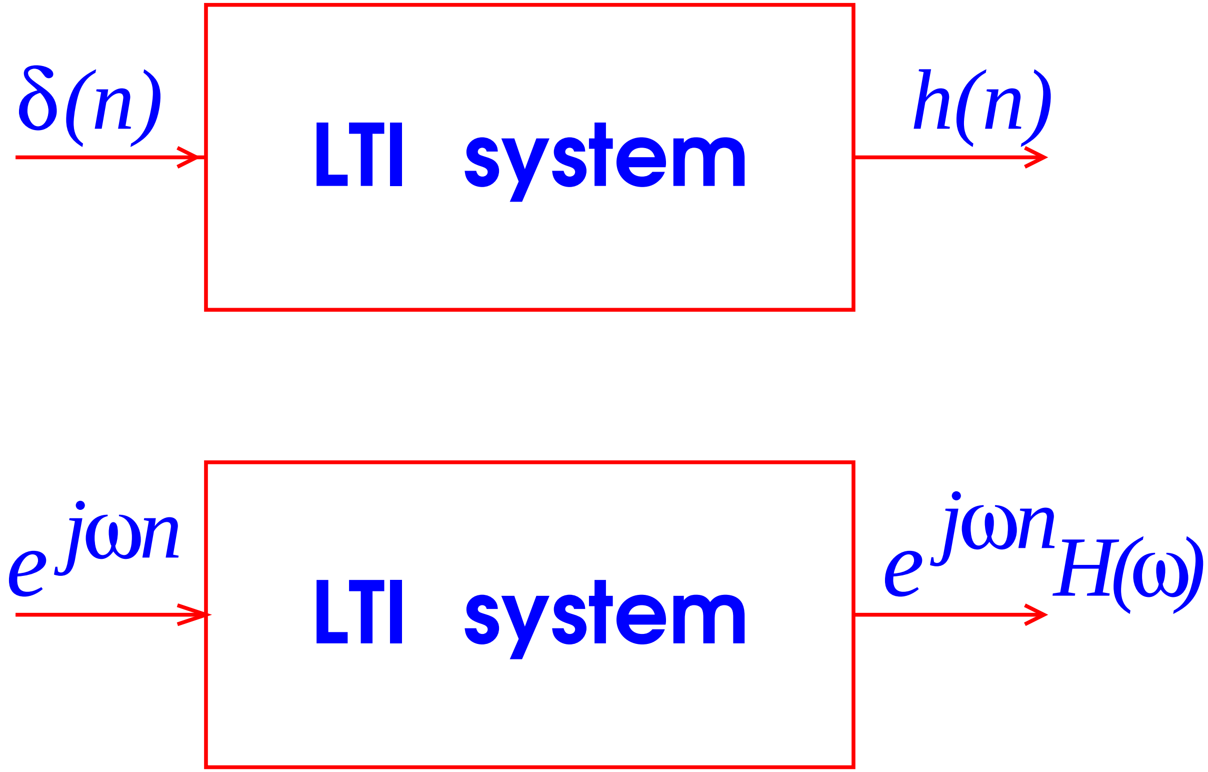
$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k}$$

is called the frequency response or the transfer function of the system.

Remarks:

- The impulse response and transfer function represent a DTFT pair $\Longrightarrow$ $H(\omega)$ is a periodic function
- The transfer function shows how different input frequency components are changed (e.g., attenuated) at system output
- This function will be very useful for the consideration of signal filtering

Interpretation of impulse and frequency responses



Example: the delay system:

$$y(n) = x(n - n_d) \quad \text{with fixed integer } n_d$$

$$x(n) = e^{j\omega n} \quad \Longrightarrow \quad y(n) = e^{j\omega(n-n_d)} \quad \Longrightarrow \quad H(\omega) = e^{-j\omega n_d}$$

Since $|H(\omega)| = 1$, this system is frequency nonselective.

Examples of frequency selective systems will be given when the filtering operation will be considered.

3.5 Elements of Sampling Theory

Question: How are CTFT and Fourier series related for periodic signals?

Consider a signal $x(t)$ with CTFT

$$X(\omega) = 2\pi\,\delta(\omega - \omega_0)$$

The signal itself is determined via the inverse relation

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega - \omega_0) e^{j\omega t}\,d\omega = e^{j\omega_0 t}$$

We know that periodic signals have line equispaced spectrum. Let $X(\omega)$ be a linear combination of impulses equally spaced in frequency:

$$X(\omega) = \sum_{n=-\infty}^{\infty} 2\pi X_n \delta(\omega - n\omega_0) \qquad (*)$$

Using inverse CTFT, i.e., applying it to each term in the sum, we obtain

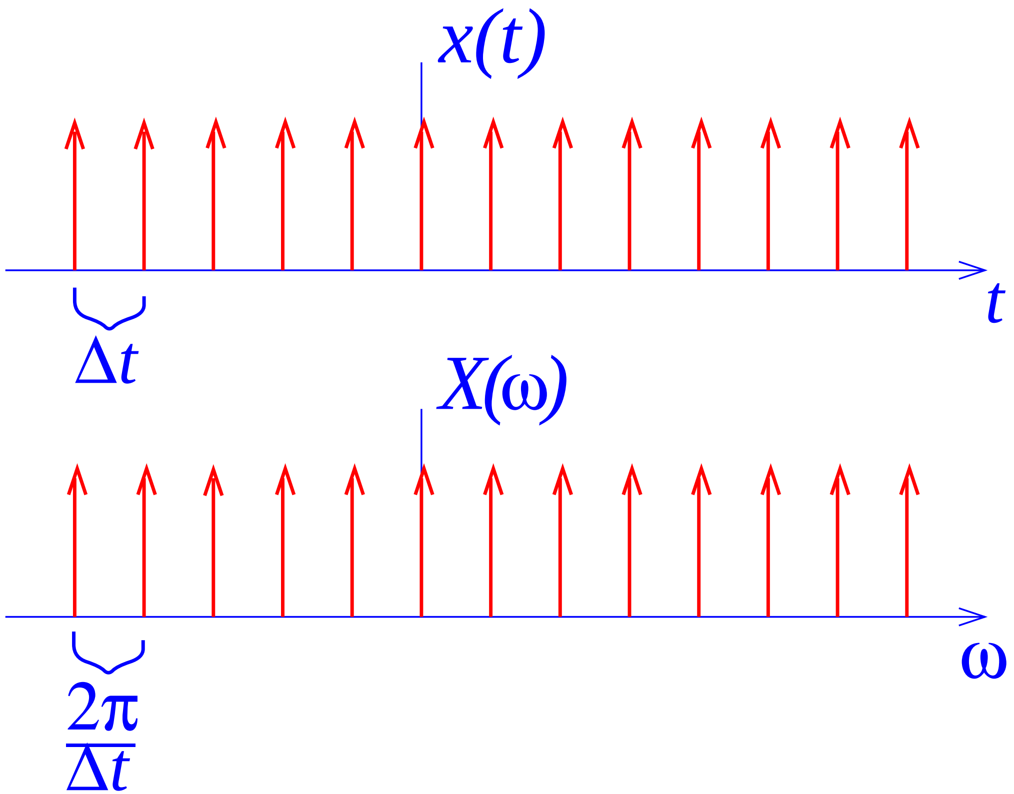
$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi X_n \delta(\omega - n\omega_0) e^{j\omega t}\, d\omega \\ &= \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \qquad \text{exactly Fourier series!} \end{aligned}$$

Result: CTFT of a periodic signal with Fourier series coefficients $\{X_n\}$ can be interpreted as a train of impulses occuring at the harmonically related frequencies with the weights $\{2\pi X_n\}$.

Remark: Periodic impulse train $(*)$ is neither absolutely nor square summable. Hence, the CTFT is introduced and understood in a limiting sense.

Result: the Fourier transform of a periodic impulse train is a periodic impulse train.

$$\sum_{k=-\infty}^{\infty} \delta(t - k\,\Delta t) \quad \leftrightarrow \quad \frac{2\pi}{\Delta t} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{\Delta t}\right)$$



Proof: The impulse train $\sum_{k=-\infty}^{\infty} \delta(t - k\,\Delta t)$ is a periodic signal with period Δt $\Longrightarrow$ applying Fourier series, we obtain, that the Fourier coefficients

$$X_n = \frac{1}{\Delta t}\int_{-\Delta t/2}^{\Delta t/2} \delta(t) e^{-j\frac{2\pi n}{\Delta t}t}\,dt = \frac{1}{\Delta t}$$

From $(*)$, we obtain

$$\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} 2\pi X_n \delta(\omega - n\underbrace{\omega_0}_{\frac{2\pi}{\Delta t}}) \\ &= \frac{2\pi}{\Delta t}\sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{\Delta t}\right) \end{aligned}$$

Result is proven.

Let

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t)$$

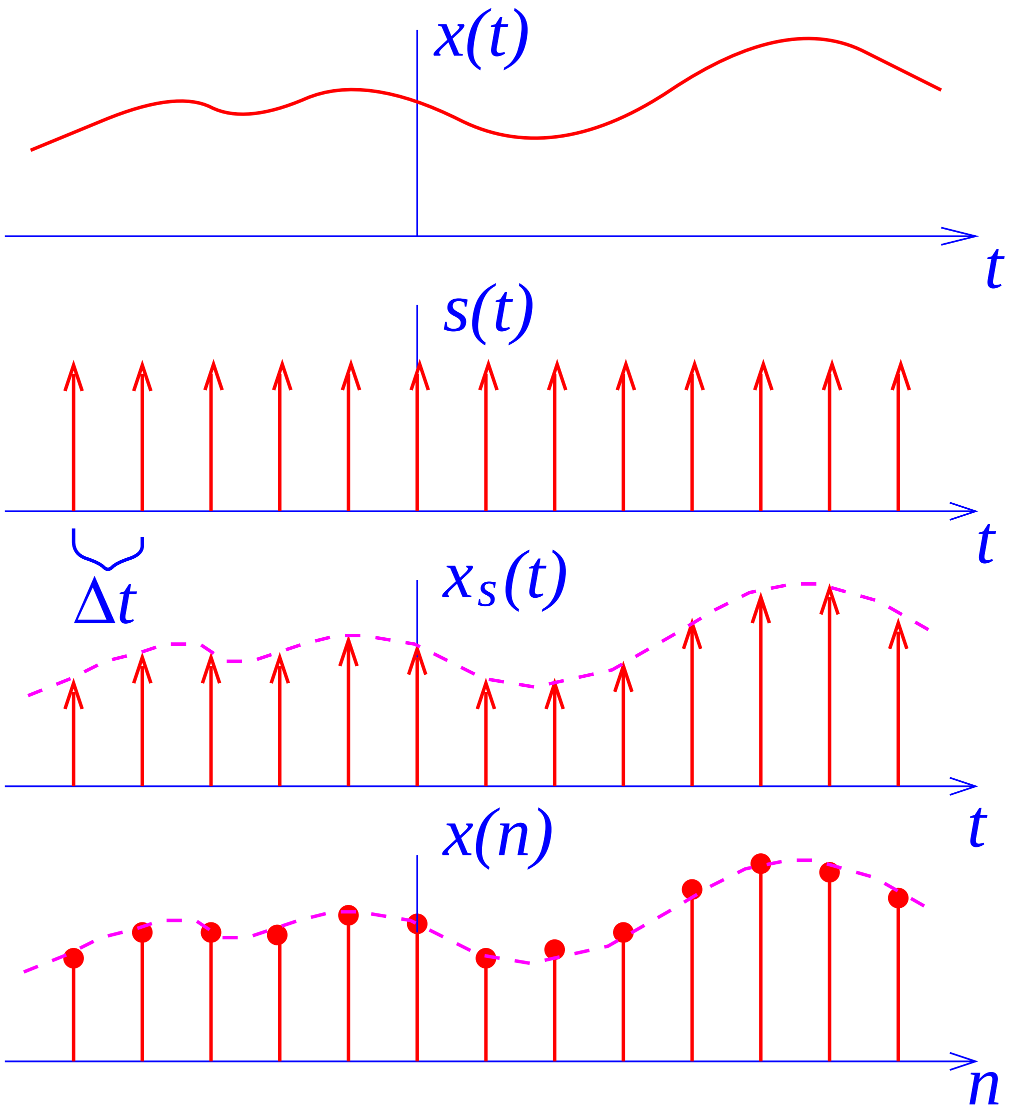
Introduce the "modulated" signal

$$x_s(t) = x(t)\, s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t)$$

Since $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$, we obtain

$$x_s(t) \;=\; \sum_{n=-\infty}^{\infty} x(n\,\Delta t)\delta(t - n\Delta t)$$

Using this "modulated" signal, we characterize the operation of sampling.



from analog to digital signal using impulse train

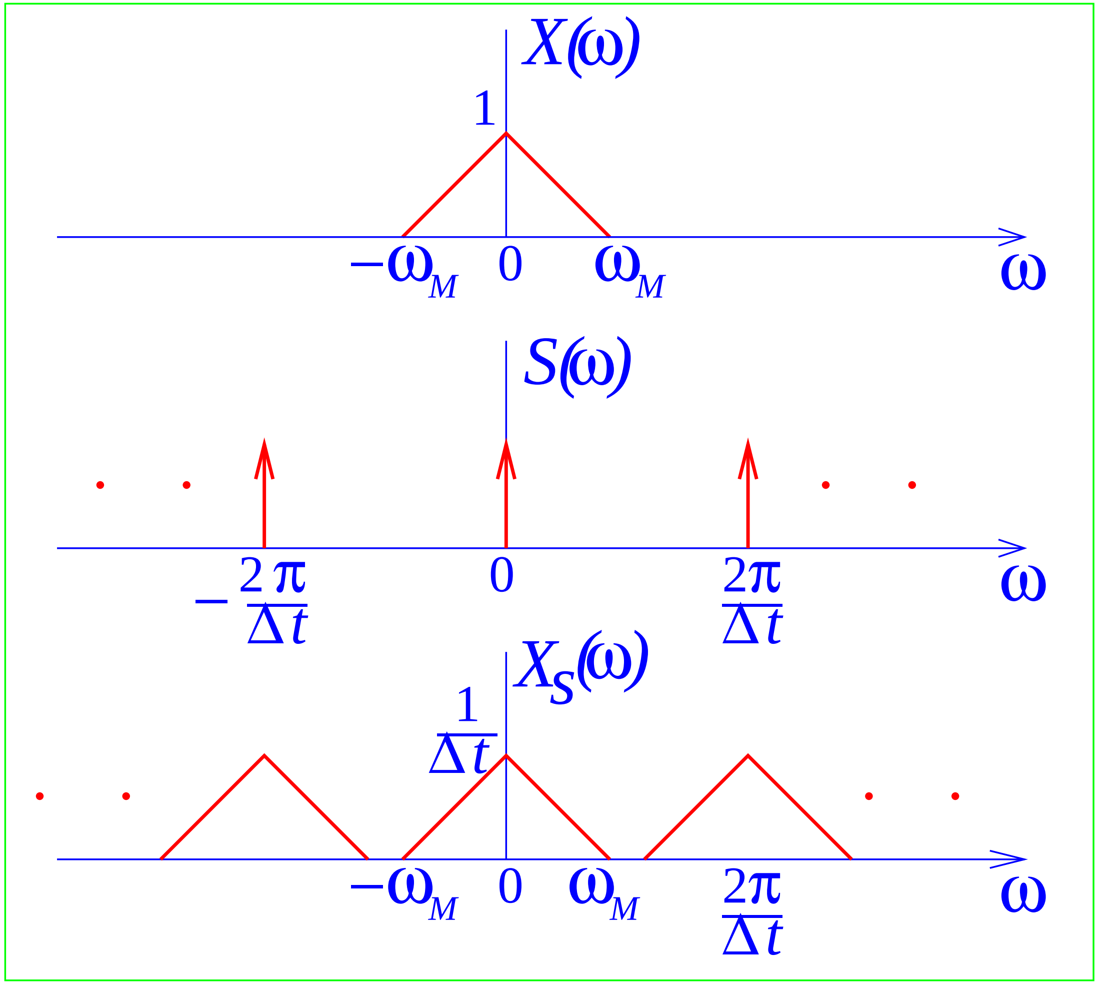
Use windowing theorem:

$$x_s(t) = x(t)\, s(t) \quad \Longrightarrow \quad X_s(\omega) = \frac{1}{2\pi} \{X(\omega)\} * \{S(\omega)\}$$

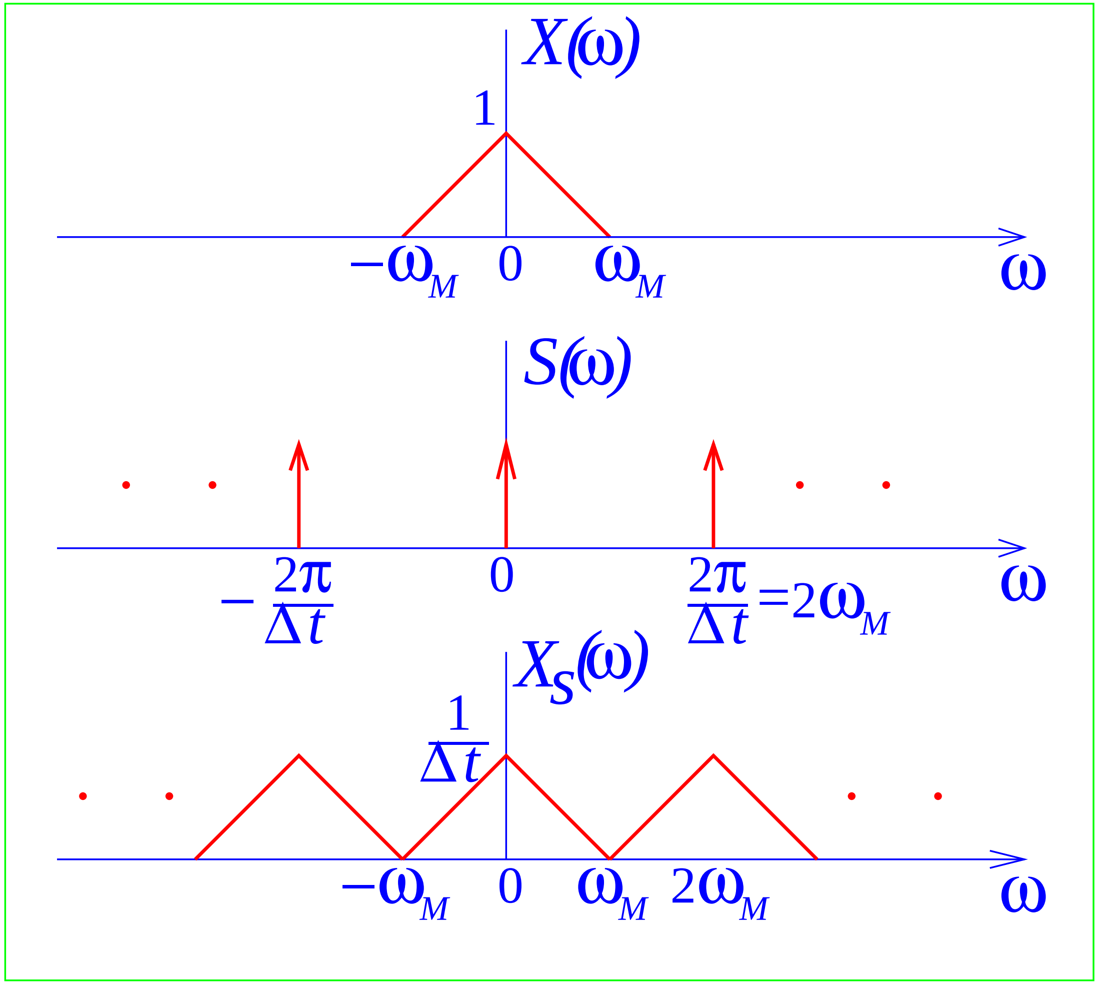
Hence,

$$\begin{aligned}
X_s(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\nu) S(\omega - \nu)\, d\nu \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\nu) \left\{ \frac{2\pi}{\Delta t} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{\Delta t} - \nu \right) \right\} \\
&= \frac{1}{\Delta t} \sum_{n=-\infty}^{\infty} X\left(\omega - \frac{2\pi n}{\Delta t} \right)
\end{aligned}$$

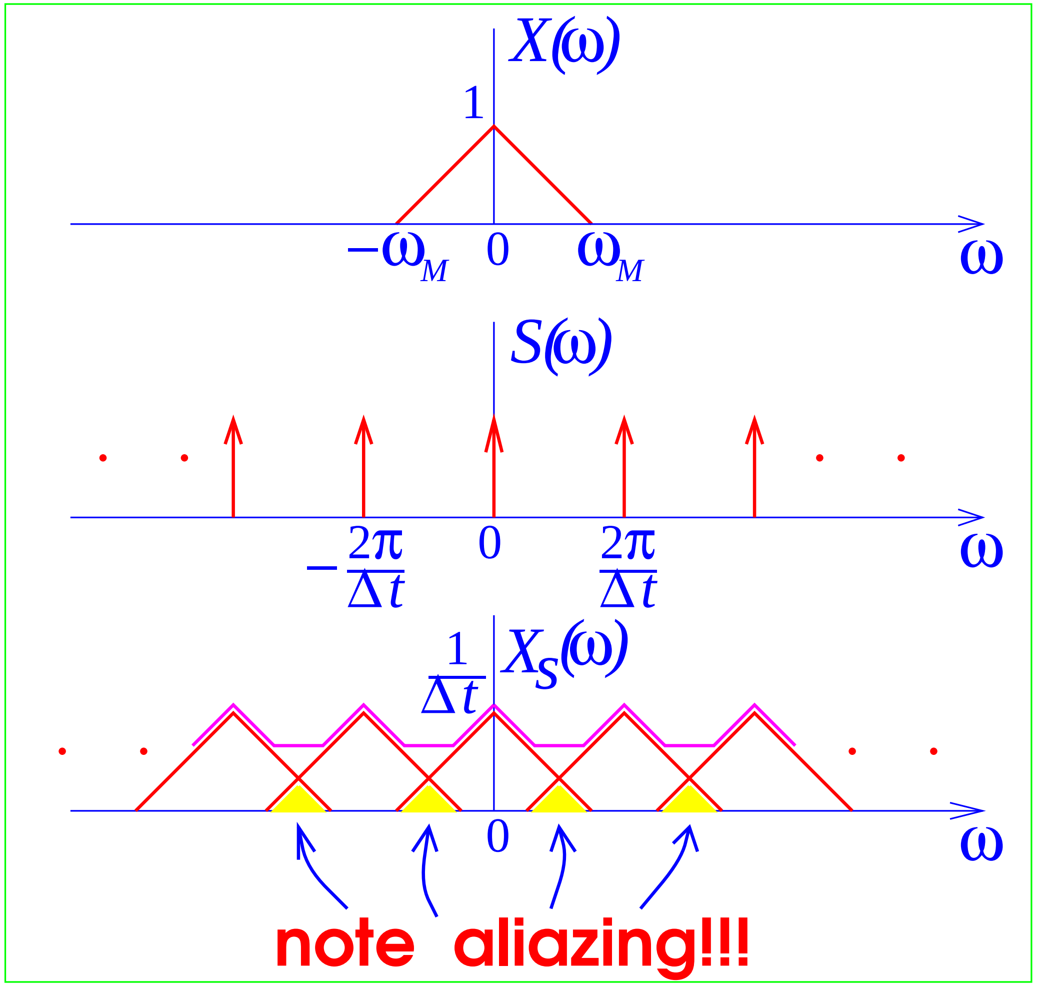
Effect in frequency domain when sampling is done in time domain faster than at Nyquist rate ($\frac{2\pi}{\Delta t} > 2\,\omega_M$)



Effect in frequency domain when sampling is done in time domain exactly at Nyquist rate ($\frac{2\pi}{\Delta t} = 2\,\omega_M$)



Effect in frequency domain when sampling is done in time domain slower than at Nyquist rate ($\frac{2\pi}{\Delta t} < 2\,\omega_M$)



Introduce a lowpass filtering operation. The spectrum of the filtered signal:

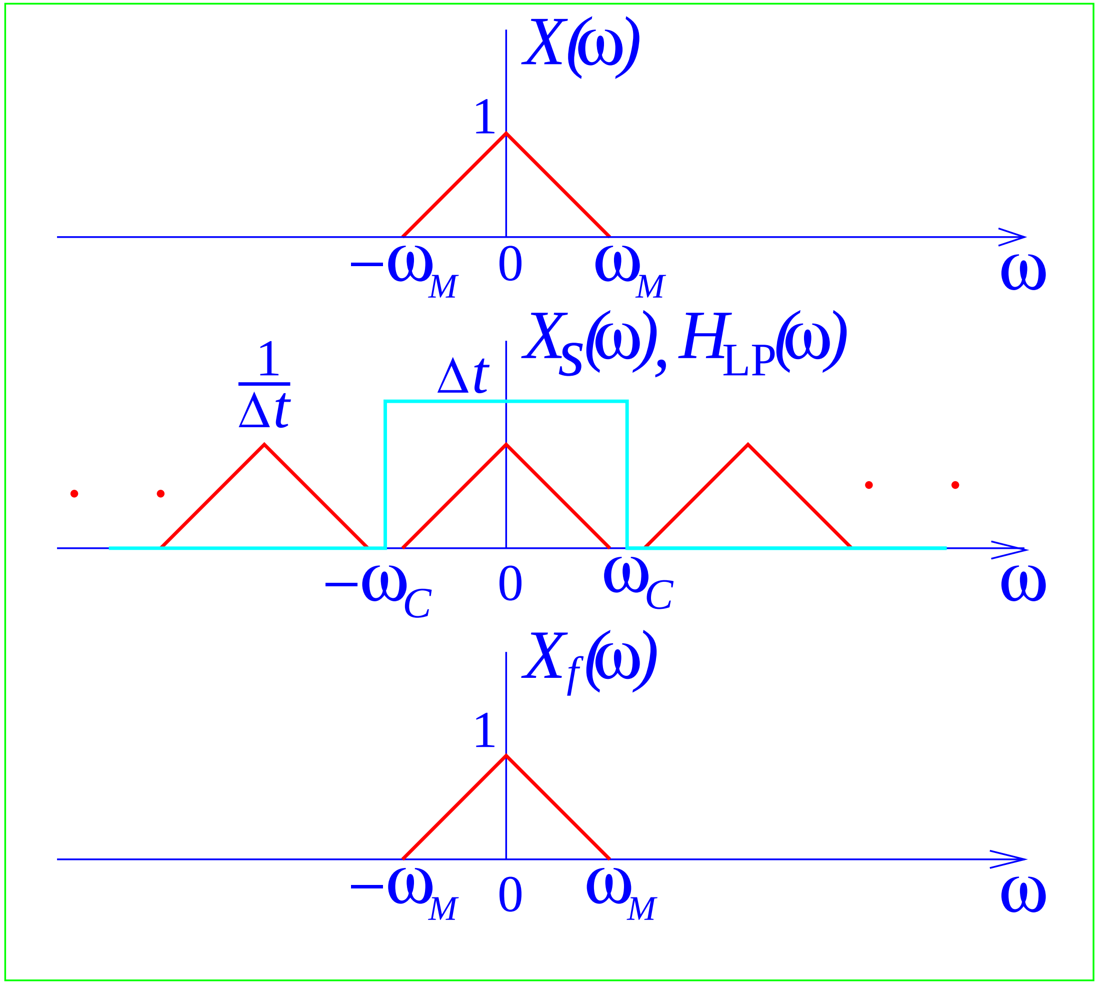
$$X_f(\omega) = H_{\mathrm{LP}}(\omega) X_s(\omega)$$

where an ideal filter transfer function

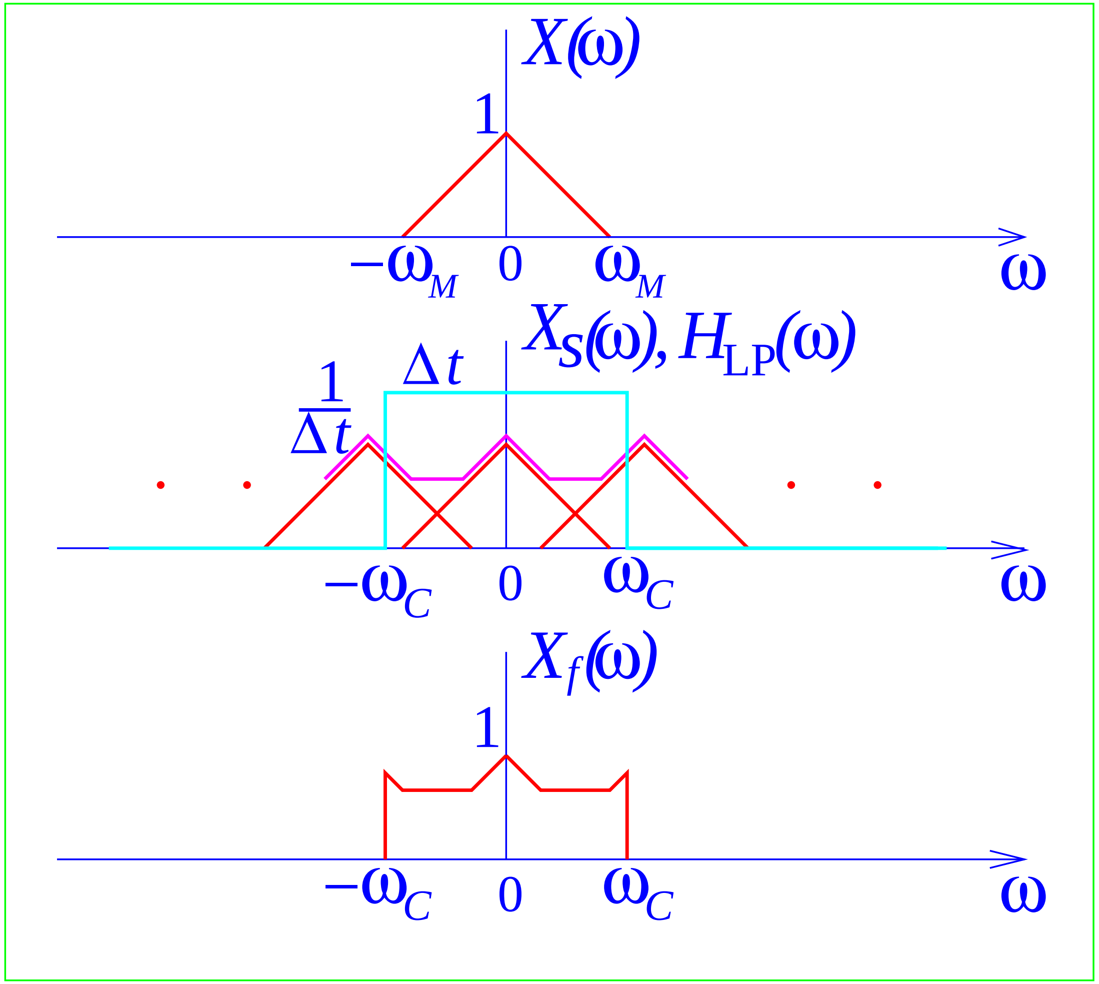
$$H_{\mathrm{LP}}(\omega) = \begin{cases} \Delta t\,, & -\omega_c \le \omega \le \omega_c \\ 0\,, & \text{otherwise} \end{cases}$$

with the cut-off frequency ω_c

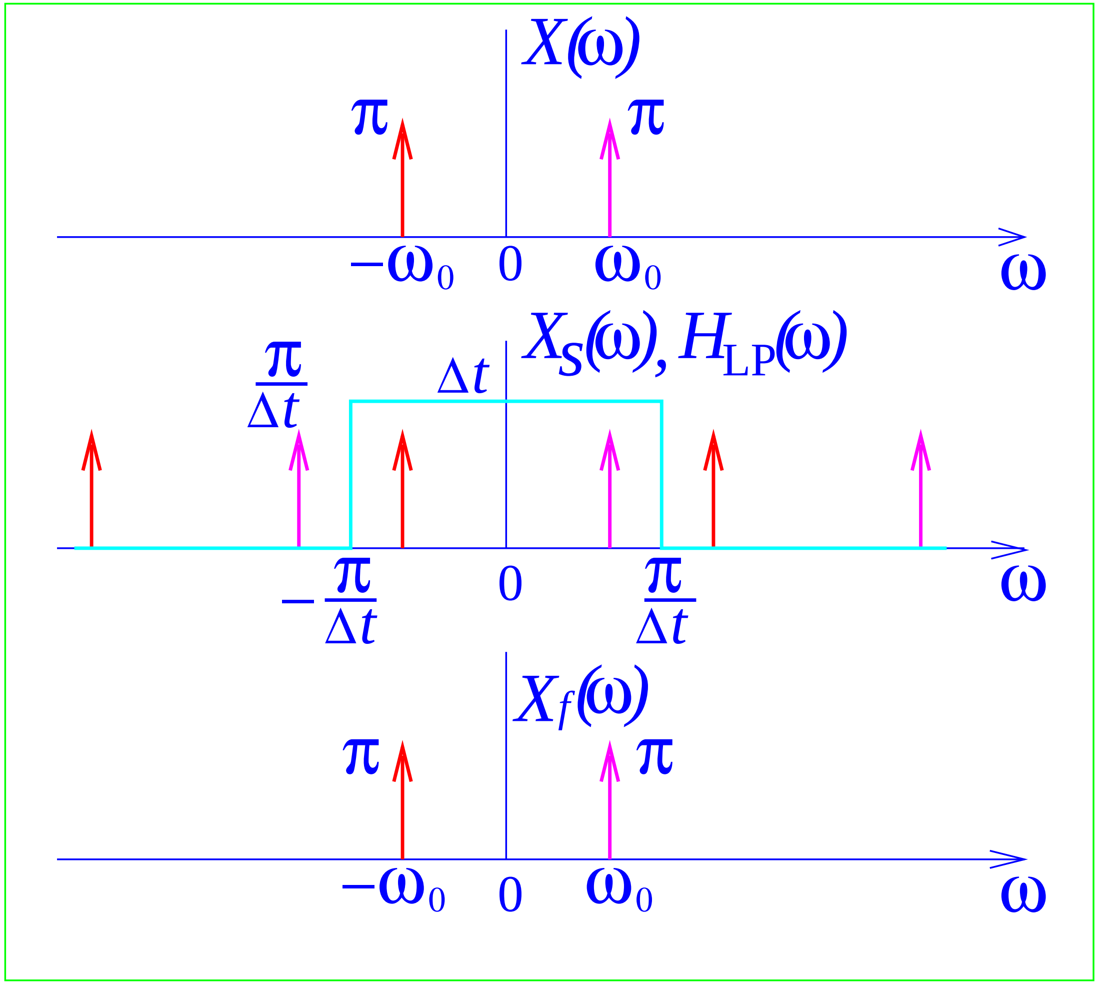
Perfect recovery of an analog signal from its samples using an ideal lowpass filter



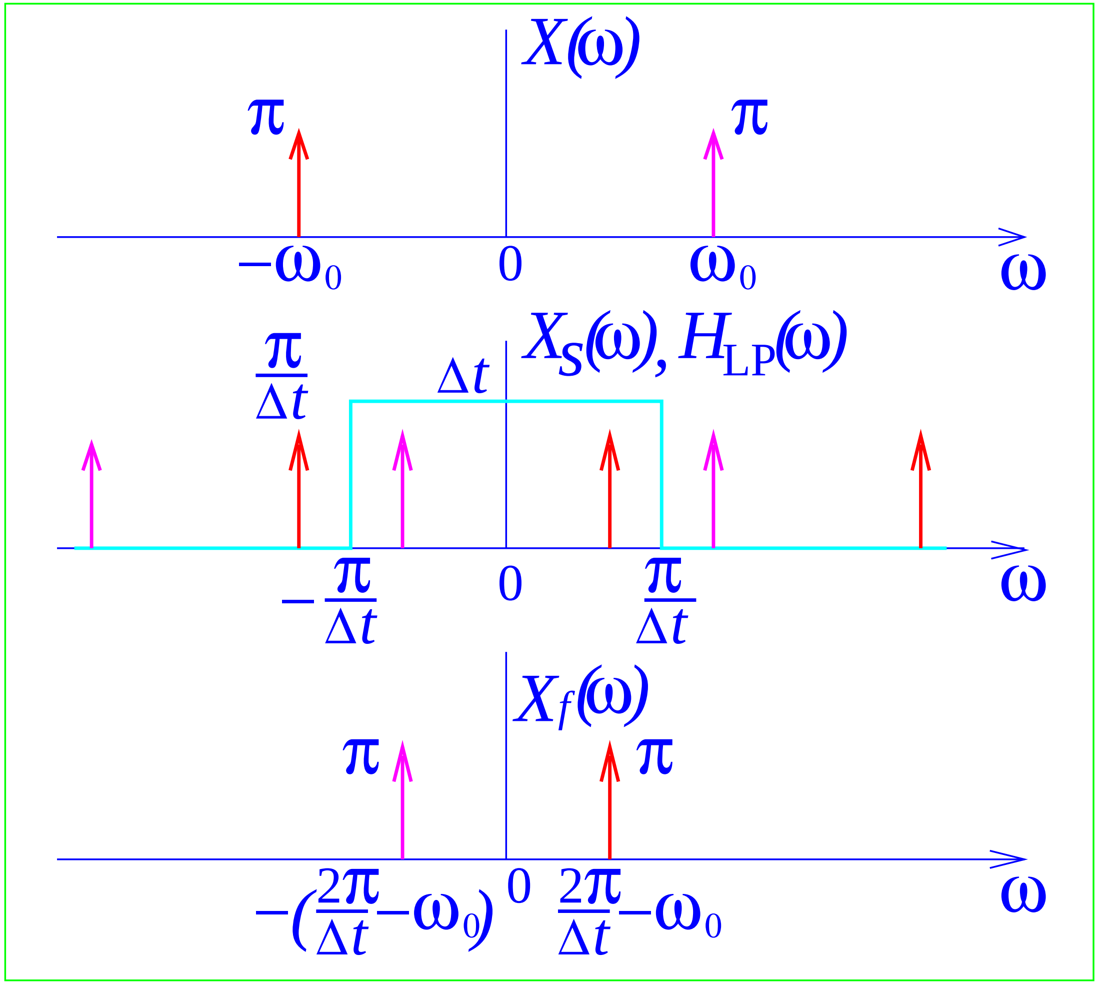
Poor recovery of an analog signal from its samples using an ideal lowpass filter



Perfect recovery of a cosine at ω_0: no aliasing because the sampling rate is higher than Nyquist rate ($\frac{2\pi}{\Delta t} > 2\,\omega_0$)



Poor recovery of a cosine at ω_0: the aliasing occurs because the sampling rate is lower than Nyquist rate ($\frac{2\pi}{\Delta t} < 2\,\omega_0$)



Now, let us determine how to reconstruct a bandlimited signal from its samples in the time domain.

Result: Having a signal sampled at a rate higher than Nyquist rate and infinite number of its discrete values, the signal can be exactly recovered as

$$x_f(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin[\pi(t - n\,\Delta t)/\Delta t]}{\pi(t - n\,\Delta t)/\Delta t}$$

This equation is referred to as the Shannon-Nyquist-Kotelnikov interpolation formula.

Proof: We have seen that the signal can be reconstructed in the frequency domain using ideal lowpass filter:

$$X_f(\omega) = X_s(\omega) H_{\mathrm{LP}}(\omega)$$

In time-domain:

$$\begin{aligned}
x_f(t) &= \{x_s(t)\} * \{h_{\mathrm{LP}}(t)\} \\
&= \left\{ \sum_{n=-\infty}^{\infty} x(n\,\Delta t)\delta(t - n\Delta t) \right\} * \{h_{\mathrm{LP}}(t)\} \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(n\,\Delta t)\delta(\tau - n\Delta t) h_{\mathrm{LP}}(t-\tau)\, d\tau \\
&= \sum_{n=-\infty}^{\infty} x(n) h_{\mathrm{LP}}(t - n\,\Delta t)
\end{aligned}$$

Ideal transfer function:

$$H_{\mathrm{LP}}(\omega) = \begin{cases} \Delta t\,, & -\frac{\pi}{\Delta t} \leq \omega \leq \frac{\pi}{\Delta t} \\ 0\,, & \text{otherwise} \end{cases}$$

Ideal impulse response

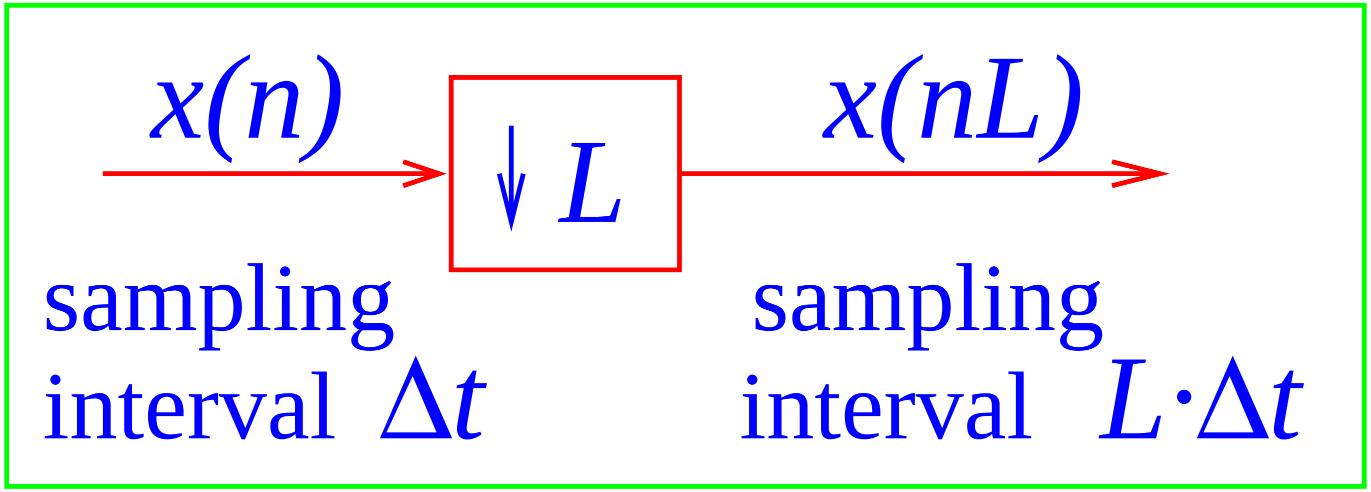
$$h_{\mathrm{LP}}(t) = \frac{\sin(\pi t/\Delta t)}{\pi t/\Delta t} \qquad (*)$$

Now, insert $(*)$ into the equation for $x_f(t)$. Result is proven.

generic digital system structure

Sampling rate reduction by an integer factor:

$$x_{\mathrm{d}}(n) = x(nL) \qquad \text{decimation, downsampling}$$



To avoid aliasing, the signal $x(n)$ should be filtered with the cutoff frequency $\omega_M < \pi/\Delta t \quad \Longrightarrow \quad$ for decimated signal, L times lower cutoff frequency $\omega_{\mathrm{d},M} < \pi/L\Delta t$ is required, so that

$$\omega_M = L \cdot \omega_{\mathrm{d},M}$$

Consider the DTFT of $x_{\mathrm{d}}(n)$:

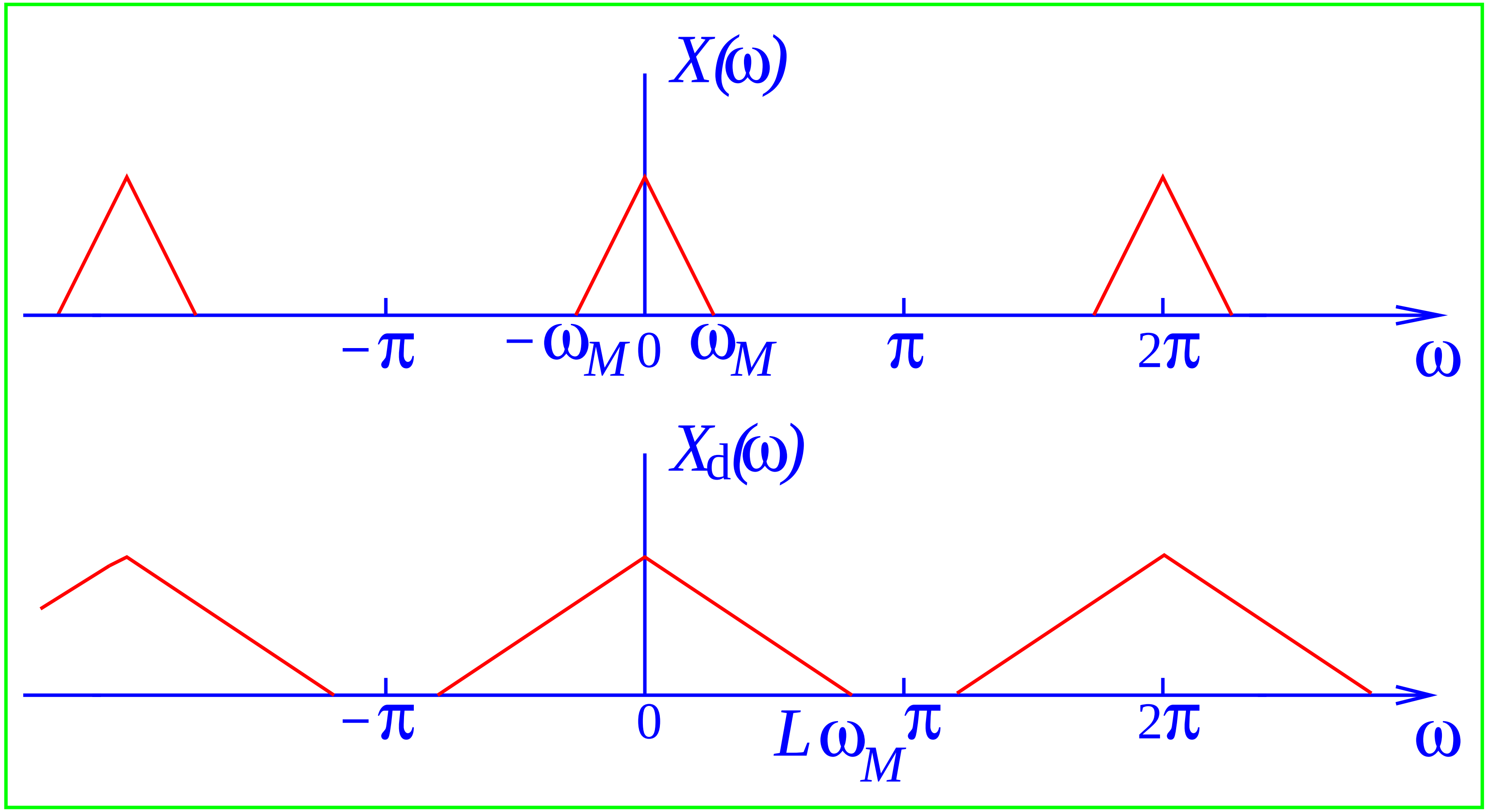
$$
\begin{aligned}
X_{\mathrm{d}}(\omega) &= \sum_{n=-\infty}^{\infty} x_{\mathrm{d}}(n)\, e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} x(\underbrace{nL}_{k})\, e^{-j\omega n} \\
&= \sum_{k=-\infty}^{\infty} x(k)\, e^{-j\omega k/L} = X(\omega/L) \qquad \Longrightarrow
\end{aligned}
$$

spectra for conventional sequence $x(n)$ and for decimated sequence $x_{\mathrm{d}}(n)$ differ only in a frequency scaling!

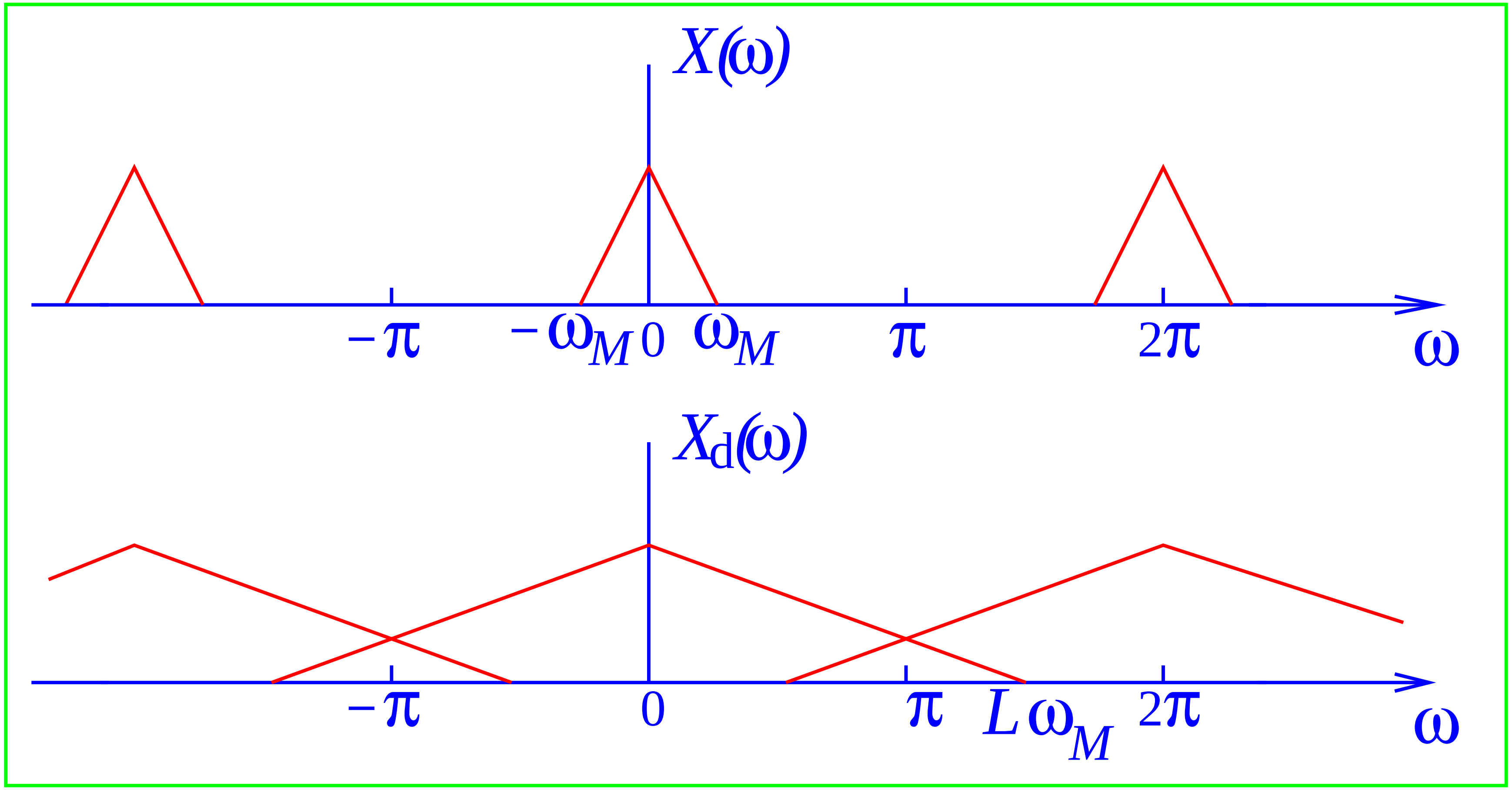
For the DTFT-frequencies, the non-aliasing conditions are:

$$
\omega_M < \pi \,, \qquad L\omega_{\mathrm{d},M} < \pi
$$

The effect of frequency scaling for the decimated sequence: no aliasing because $L\omega_M < \pi$



The effect of frequency scaling for the decimated sequence: the aliasing occurs because $L\omega_M > \pi$



Increasing the sampling rate by an integer factor:

$$x(n) = x(n\,\Delta t) \longrightarrow x_{\mathrm{i}} = x(n\,\Delta t')\,,$$
$$\Delta t' = \Delta t/L \qquad \text{interpolation, upsampling}$$

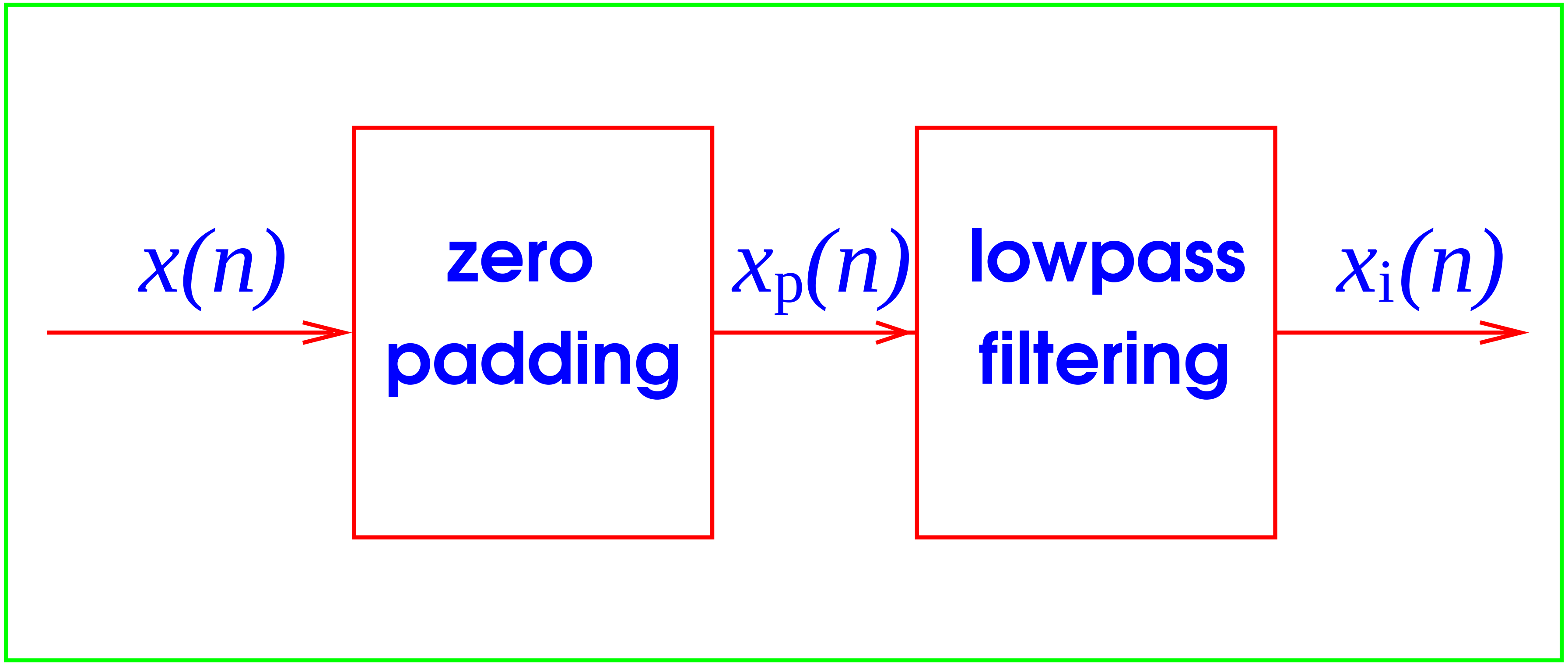
Interpolation formula can be used:

$$x_{\mathrm{i}}(n) = \sum_{k=-\infty}^{\infty} x(k)\frac{\sin[\pi(n-kL)/L]}{\pi(n-kL)/L} = \sum_{m=-\infty}^{\infty} x_{\mathrm{p}}(m)\frac{\sin[\pi(n-m)/L]}{\pi(n-m)/L}$$

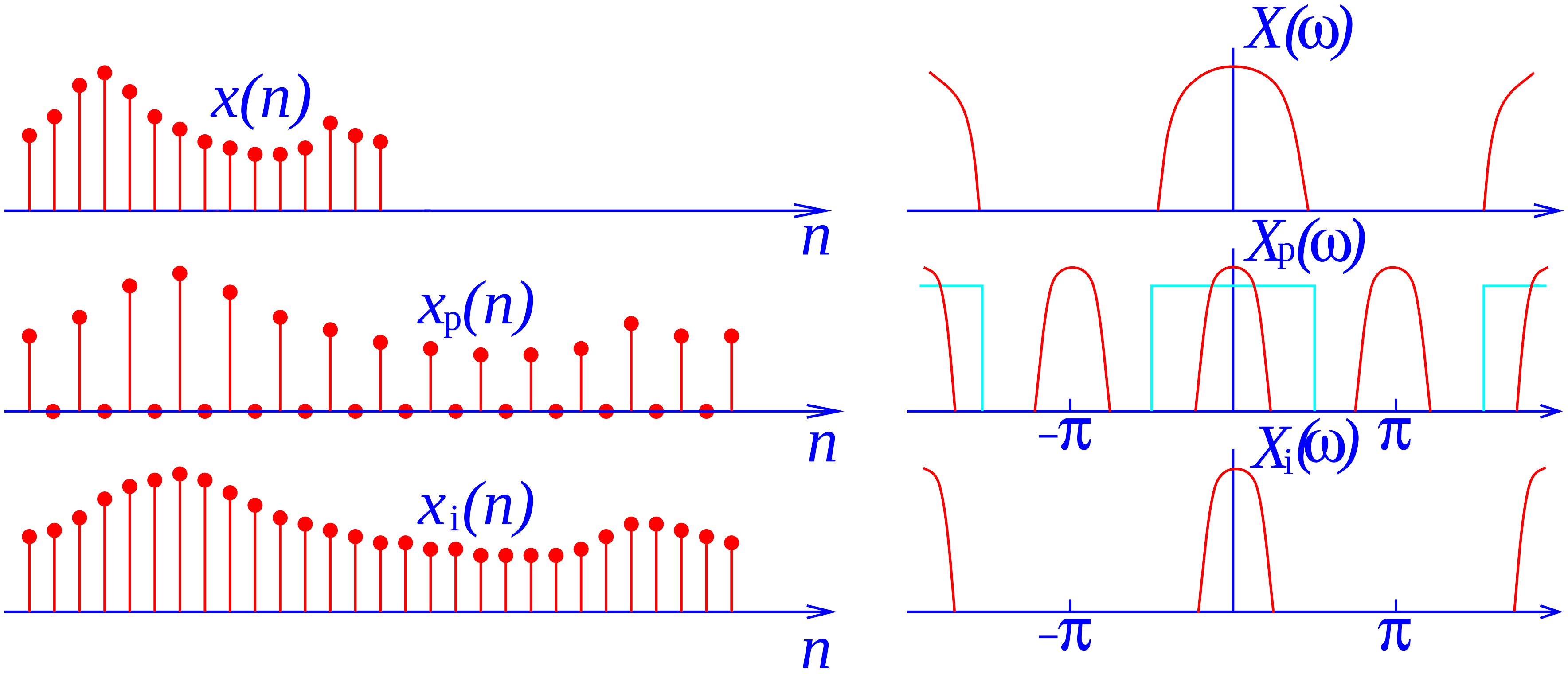
$$\{x_{\mathrm{p}}(m)\} = \{\ldots, x(-1), \underbrace{0,\ldots,0}_{L-1 \text{ points}}, x(0), \underbrace{0,\ldots,0}_{L-1 \text{ points}}, x(1), \underbrace{0,\ldots,0}_{L-1 \text{ points}}, x(2), \ldots\}$$

$\Longrightarrow$ the upsampling can be done by means of filtering of the "zero-padded" signal $x_{\mathrm{p}}(m)$ using the ideal lowpass filter with the cutoff π/L!

Upsampling system



Associated sequencies and spectra for upsampling with the factor $L = 2$



4 z-TRANSFORM

4.1 Definition and the Regions of Convergence

Consider a continuous-time LTI system with $x(t) = e^{st}$:

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}\,d\tau \\ &= e^{st}\underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}\,d\tau}_{H(s)} = H(s)e^{st} \qquad \Longrightarrow \end{aligned}$$

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt\,, \quad s = \sigma + j\omega \quad \text{Laplace transform}$$

The Laplace transform is extremely useful tool for continuous-time LTI system analysis. What about discrete-time LTI system analysis?

Let the system input is a complex exponential signal:

$$x(n) = z^n \quad \Longrightarrow \quad y(n) = H(z)\, z^n \ ,$$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n} \qquad \text{transfer function} \quad \Longrightarrow$$

we can introduce

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad z\text{—transform}$$

Relationship between the z-transform and DTFT: substitute $z = re^{j\omega}$:

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x(n) \left(re^{j\omega}\right)^{-n} \\ &= \sum_{n=-\infty}^{\infty} \{x(n) r^{-n}\} e^{-j\omega n} \\ &= \mathcal{F}\{x(n) r^{-n}\} \qquad \Longrightarrow \end{aligned}$$

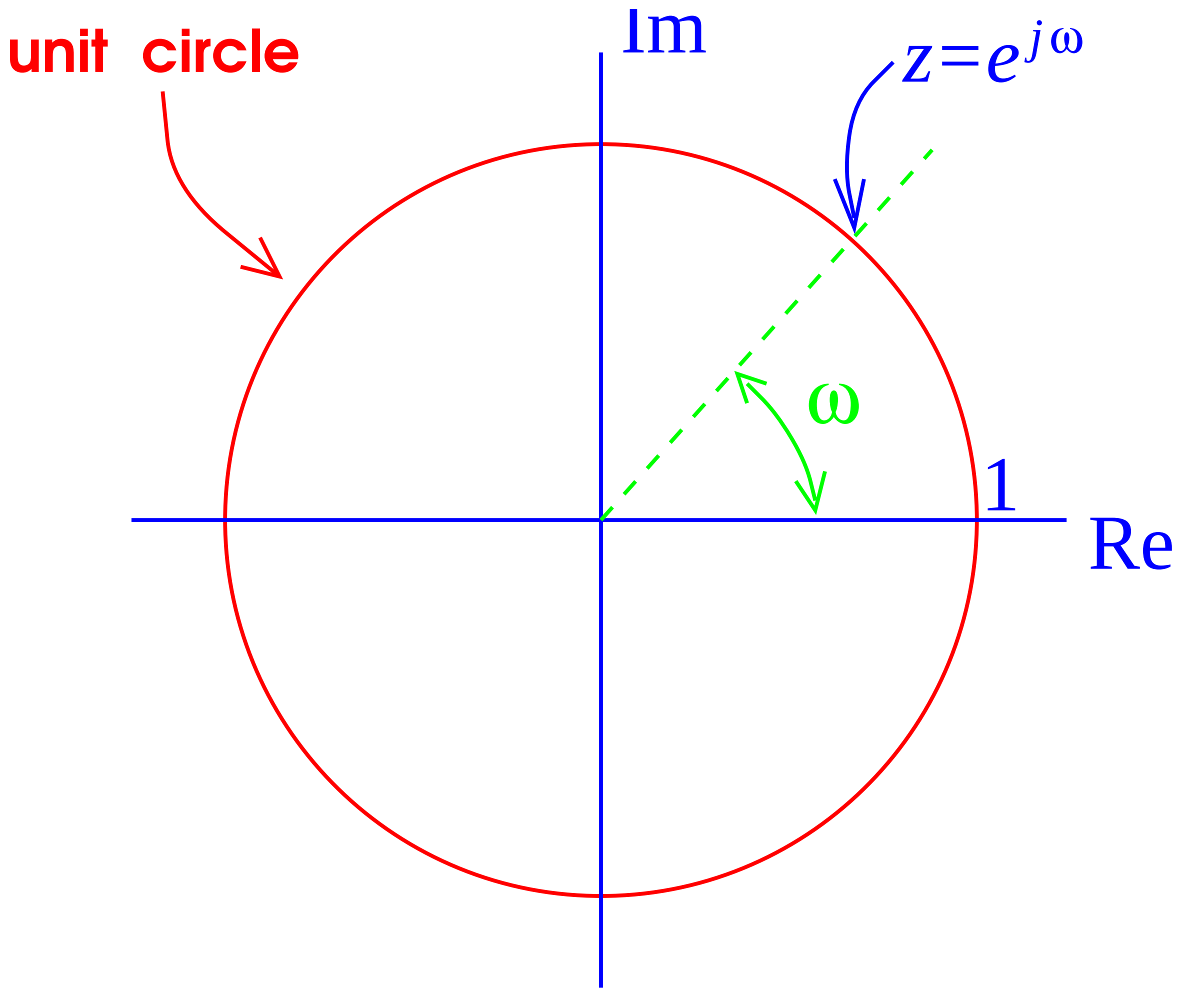
z-transform of an arbitrary sequence $x(n)$ is equivalent to DTFT of the exponentially weighted sequence $x(n)r^{-n}$.

If $r = 1$ then

$$X(z)\Big|_{z=e^{j\omega}} = X(\omega) = \mathcal{F}\{x(n)\} \qquad \Longrightarrow$$

DTFT corresponds to the particular case of z-transform with $|z| = 1$!

Complex z-plane. The z-transform reduces to the DTFT for values of z on the unit circle



Question: When does the z-transform converge?

From the relationship between DTFT and z-transforms it follows that in the general case of finite-energy signals, the z-transform does not converge for all values of z $\Longrightarrow$ there is a range of values of z (referred to as the Region Of Convergence (ROC)) for which $|X(z)| < \infty$.

Example: the z-transform of the signal $x(n) = a^n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

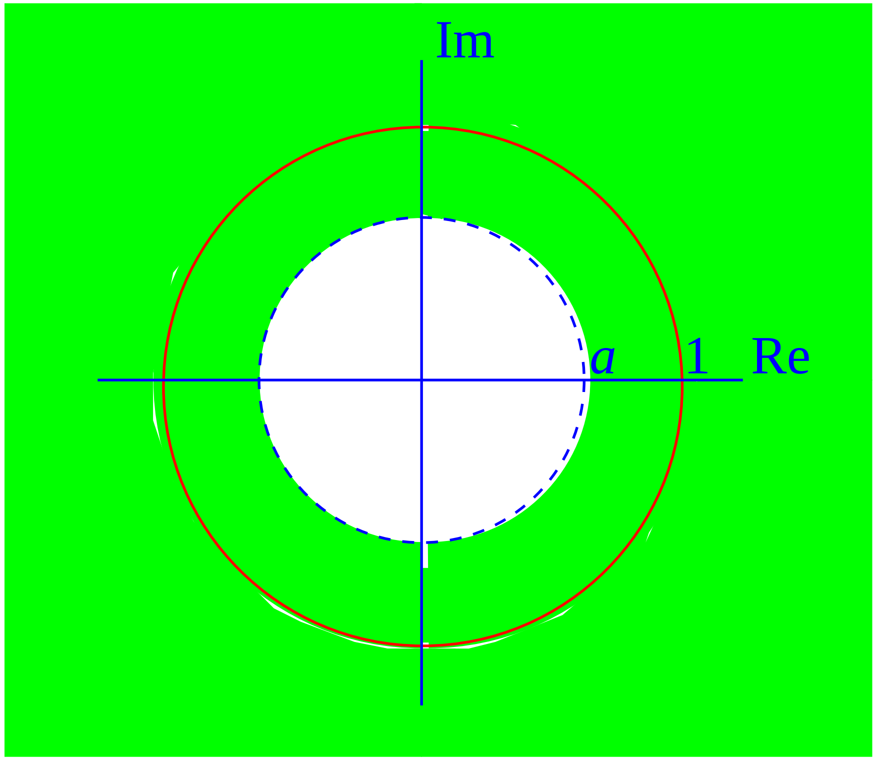
For convergence, we require that

$$\sum_{n=-\infty}^{\infty} |az^{-1}|^n < \infty$$

Recall that

$$\sum_{n=-\infty}^{\infty} |c|^n = \begin{cases} \frac{1}{1-|c|}, & |c| < 1 \\ \infty, & |c| \geq 1 \end{cases} \qquad \Longrightarrow$$

the ROC is determined by $|z| > |a|$

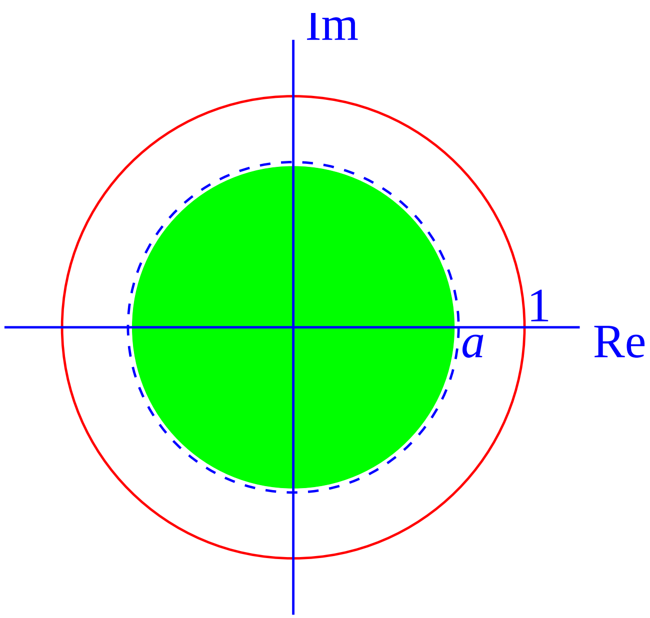


Example: now, let the signal be $x(n) = -a^n u(-n-1)$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

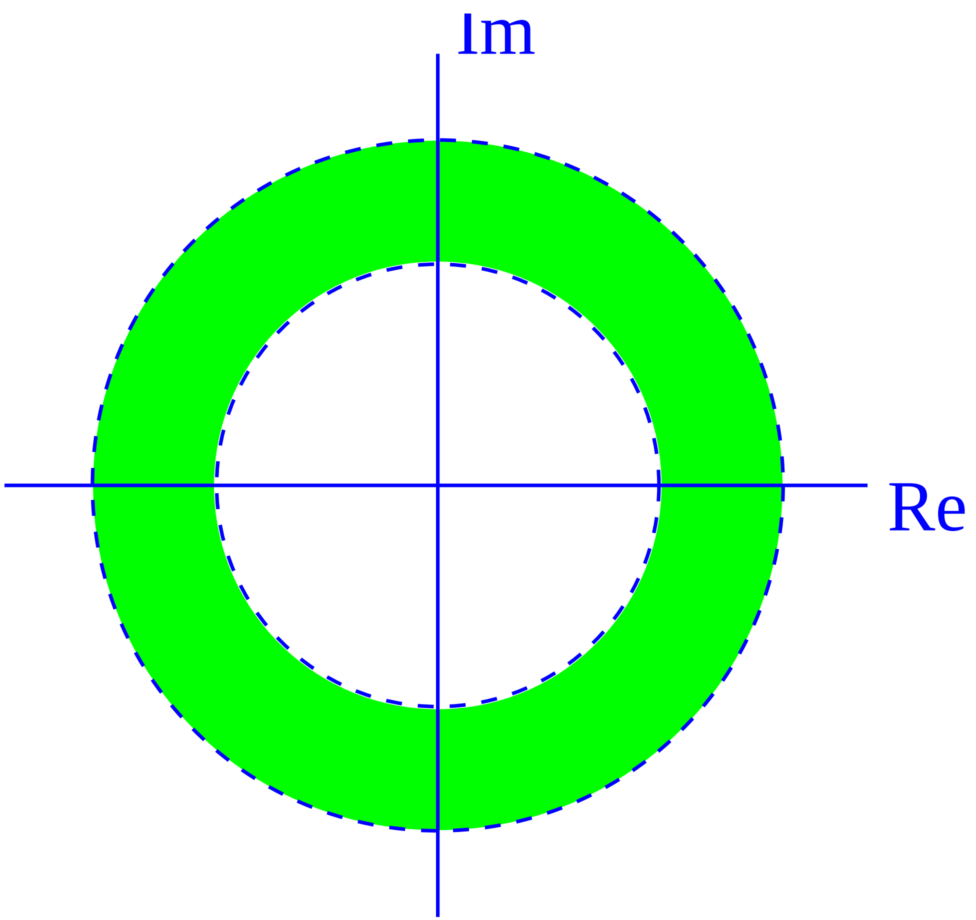
$$= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} \left(a^{-1} z\right)^n \quad \Longrightarrow$$

the ROC is determined by $|z| < |a|$



4.2 Properties of the ROC

Property 1: The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin.



Property 2: The ROC does not contain any poles.

Property 3: If $x(n)$ is of finite duration, then the ROC is the entire z-plane except possibly $z = 0$ and/or $z = \infty$.

$$X(z) = \sum_{n=N_1}^{N_2} x(n) z^{-n} \qquad \text{finite duration signal}$$

Particular cases:

- if $N_1 < 0$ and $N_2 > 0$ then the ROC does not include $z = 0$ and $z = \infty$
- if $N_1 \geq 0$ then the ROC includes $z = \infty$, but does not include $z = 0$
- if $N_2 \leq 0$ then the ROC includes $z = 0$, but does not include $z = \infty$

Property 4: If $x(n)$ is a right-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all finite values of z for which $|z| > r_0$ will also be in the ROC.

$$X(z) = \sum_{n=N_1}^{\infty} x(n) z^{-n} \qquad \text{right} - \text{sided sequence}$$

Particular cases:

- if $N_1 < 0$ then the ROC does not include $z = \infty$
- if $N_1 \geq 0$ then the ROC includes $z = \infty$

Property 5: If $x(n)$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then all values of z for which $0 < |z| < r_0$ will also be in the ROC.

$$X(z) = \sum_{n=-\infty}^{N_2} x(n) z^{-n} \qquad \text{left} - \text{sided sequence}$$

Particular cases:

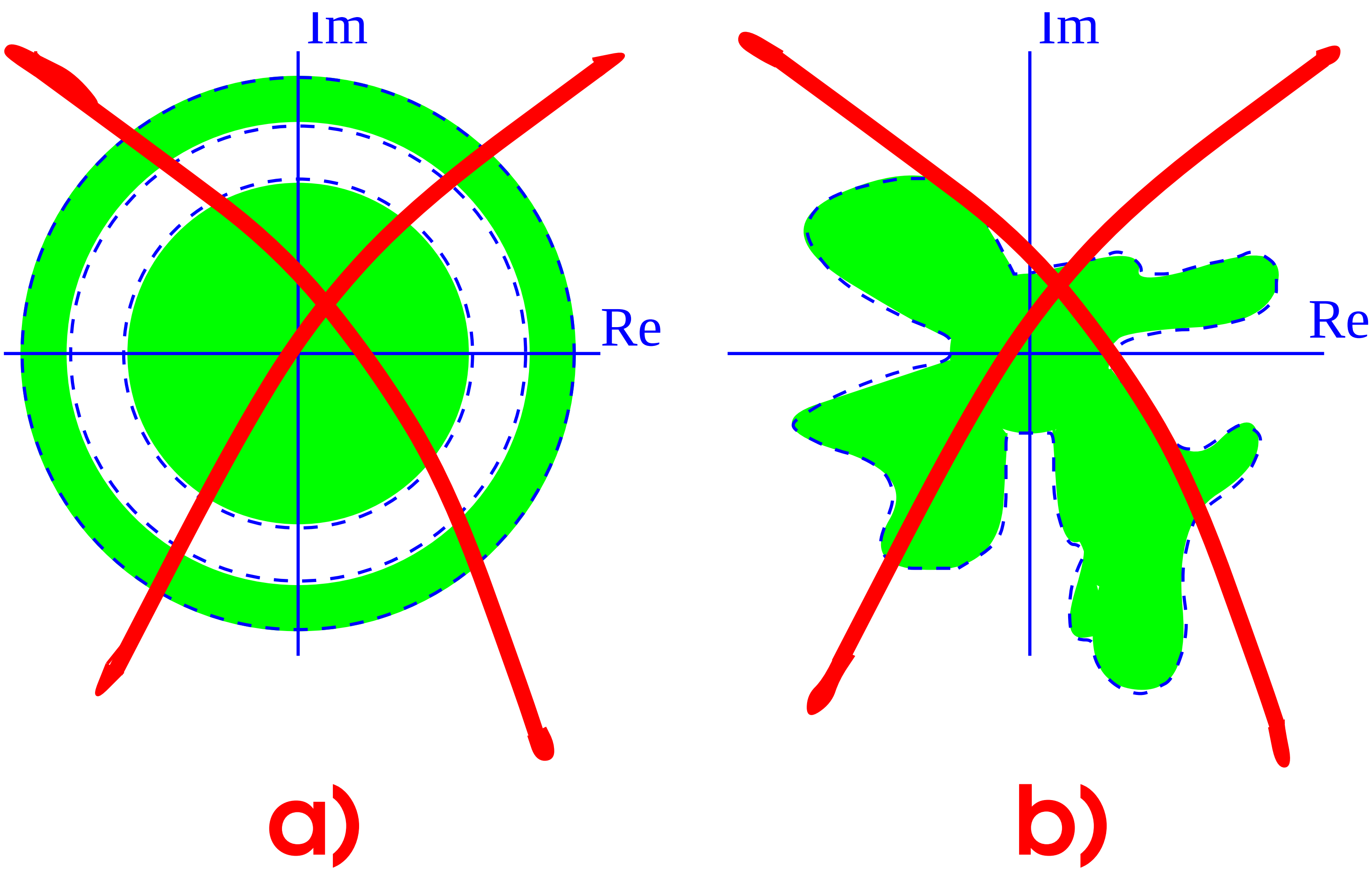
- if $N_2 > 0$ then the ROC does not include $z = 0$
- if $N_2 \leq 0$ then the ROC includes $z = 0$

Property 6: If $x(n)$ is a two-sided sequence, and if the circle $|z| = r_0$ is in the ROC, then the ROC will be a ring in the z-plane that includes the circle $|z| = r_0$.

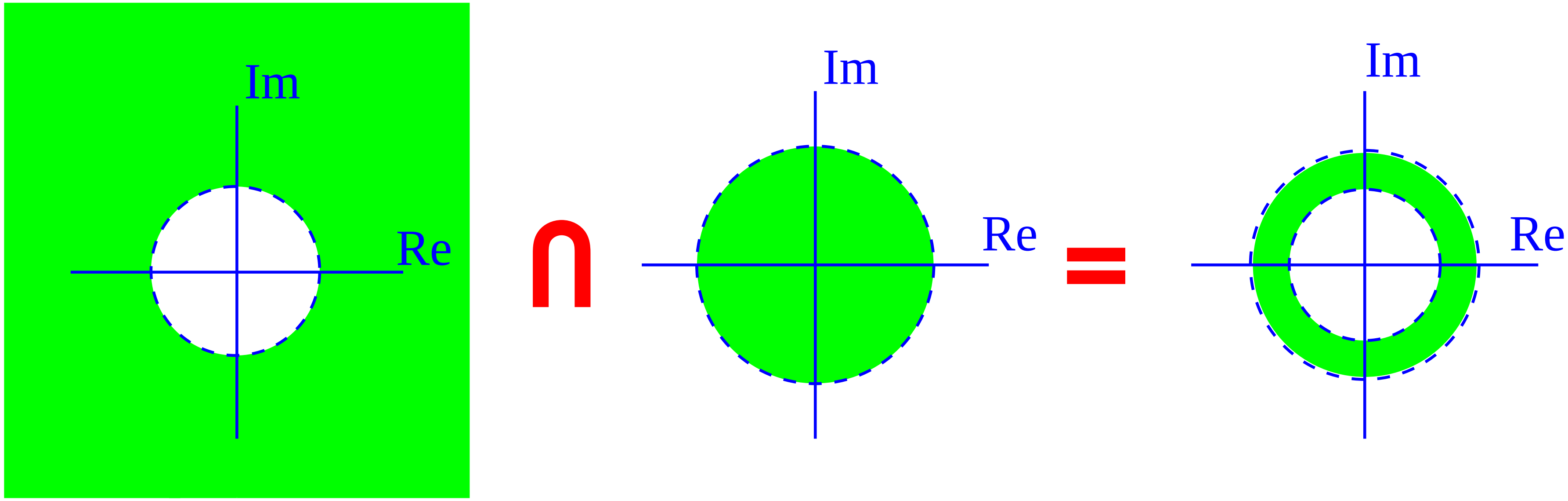
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad \text{two} - \text{sided sequence}$$

Any two-sided sequence can be represented as a direct sum of a right-sided and left-sided sequencies $\Longrightarrow$ the ROC of this composite signal will be the intersection of the ROC's of the components.

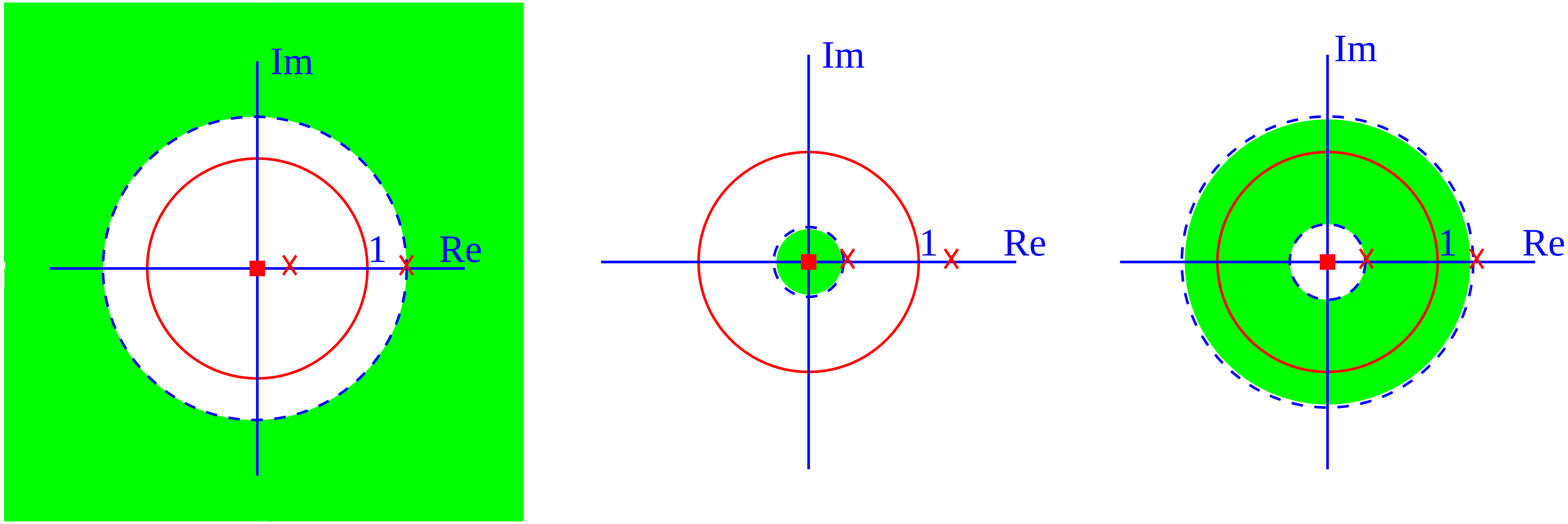
To property 1: a) the ROC must be a connected region, b) the ROC cannot be nonsymmetric



To property 6: intersection of the ROC's of right-sided and left-sided sequencies



To properties 2, 4, 5, 6: pole-zero pattern and three possible ROC's that correspond to $X(z) = \left[(1-\frac{1}{3}z^{-1})(1-1.3z^{-1})\right]^{-1}$



a) right-sided b) left-sided c) two-sided

4.3 The Inverse z-Transform

We obtained that

$$X(z)\Big|_{z=re^{j\omega}} = \mathcal{F}\{x(n)r^{-n}\}$$

Applying the inverse DTFT, we get

$$\begin{aligned} x(n) &= r^n \mathcal{F}^{-1}\{X(re^{j\omega})\} \\ &= r^n \frac{1}{2\pi}\int_{-\pi}^{\pi} X(re^{j\omega})e^{j\omega n}\, d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\underbrace{re^{j\omega}}_{z})(\underbrace{re^{j\omega}}_{z})^n\, d\omega \\ &= \frac{1}{2\pi j}\oint X(z)z^{n-1}\, dz \qquad \longleftarrow \qquad dz = jre^{j\omega}\, d\omega \end{aligned}$$

Remarks on inverse z-transform:

- $\oint \cdots dz$ denotes integration around a closed circular contour centered at the origin and having the radius r
- the value of r must be chosen so that the contour of integration $|z| = r$ belongs to the ROC
- contour integration in the complex plane may be a complicated task
- simpler alternative procedures exist for obtaining the sequence from its z-transform

4.4 Alternative Methods for Inverse z-Transform

Inspection method: consists simply of becoming familiar with (or recognizing "by inspection") certain transform pairs.

Examples:

$$a^n u(n) \leftrightarrow \frac{1}{1 - az^{-1}}\,, \quad |z| > |a|$$

$$\delta(n-m) \leftrightarrow z^{-m}\,, \quad z \neq 0 \text{ if } m > 0\,, \; z \neq \infty \text{ if } m < 0$$

$$\sin(\omega n)\, u(n) \leftrightarrow \frac{[\sin \omega] z^{-1}}{1 - [2\cos\omega] z^{-1} + z^{-2}}\,, \quad |z| > 1$$

Extendeded inspection method: consists of expressing a complicated $X(z)$ as a sum of simpler terms and then applying to each term the inspection method.

Example:

$$X(z) = z^2(1-0.5z^{-1})(1+z^{-1})(1-z^{-1})$$
$$= z^2 - 0.5z - 1 + 0.5z^{-1} \quad \Longrightarrow$$

$$x(n) = \delta(n+2) - 0.5\delta(n+1) - \delta(n) + 0.5\delta(n-1)$$

4.5 Properties of z-Transform

Linearity:

$$\text{If} \quad X(z) = \mathcal{Z}\{x(n)\} \quad \text{and} \quad Y(z) = \mathcal{Z}\{y(n)\}$$

$$\text{then} \quad a\,X(z) + b\,Y(z) = a\,\mathcal{Z}\{x(n)\} + b\,\mathcal{Z}\{y(n)\}$$

$$\text{Also if} \quad x(n) = \mathcal{Z}^{-1}\{X(z)\} \quad \text{and} \quad y(n) = \mathcal{Z}^{-1}\{Y(z)\}$$

$$\text{then} \quad a\,x(n) + b\,y(n) = a\,\mathcal{Z}^{-1}\{X(z)\} + b\,\mathcal{Z}^{-1}\{Y(z)\}$$

$$\text{with} \quad \mathrm{ROC} = \mathrm{ROC}_x \cap \mathrm{ROC}_y$$

Time shifting:

If $\quad X(z) = \mathcal{Z}\{x(n)\} \quad$ then $\quad X(z)z^{-m} = \mathcal{Z}\{x(n-m)\}$

Also if $\quad x(n) = \mathcal{Z}^{-1}\{X(z)\}$ then $x(n-m) = \mathcal{Z}^{-1}\{X(z)z^{-m}\}$

with $\mathrm{ROC} = \mathrm{ROC}_x$ except for possible

addition/deletion of $z = 0$ or $z = \infty$

Example: for $|z| > 0.25$, consider

$$X(z) = \frac{1}{z - 0.25} = z^{-1}\left(\frac{1}{1 - 0.25z^{-1}}\right)$$

$$= z^{-1}\mathcal{Z}\{0.25^n u(n)\} \quad \Longrightarrow \quad x(n) = 0.25^{n-1}u(n-1)$$

Multiplication by exponential sequence:

$$\text{If} \quad X(z) = \mathcal{Z}\{x(n)\} \quad \text{then} \quad X(z/z_0) = \mathcal{Z}\{x(n)z_0^n\}$$

$$\text{Also if} \quad x(n) = \mathcal{Z}^{-1}\{X(z)\} \ \text{then} \ x(n)z_0^n = \mathcal{Z}^{-1}\{X(z/z_0)\}$$

$$\text{with} \ \ \text{ROC} = |z_0| \, \text{ROC}_x$$

Differentiation of $X(z)$:

$$\text{If} \quad X(z) = \mathcal{Z}\{x(n)\} \quad \text{then} \quad -z\,\frac{dX(z)}{dz} = \mathcal{Z}\{n\,x(n)\}$$

$$\text{Also if} \quad x(n) = \mathcal{Z}^{-1}\{X(z)\} \ \text{then} \ n\,x(n) = \mathcal{Z}^{-1}\{-z\,\frac{dX(z))}{dz}\}$$

$$\text{with} \ \ \text{ROC} = \text{ROC}_x$$

Example: Starting with the known transform pair

$$u(n) \quad \leftrightarrow \quad \frac{1}{1-z^{-1}} \, , \qquad |z| > 1$$

determine $X(z)$ of

$$x(n) = 2r^n \cos(\omega n) = \left(re^{j\omega}\right)^n u(n) + \left(re^{-j\omega}\right)^n u(n)$$

Using the exponential multiplication property, we have

$$\left(re^{j\omega}\right)^n u(n) \;\leftrightarrow\; \frac{1}{1-z^{-1}re^{j\omega}} \, , \quad |z| > |r|$$

$$\left(re^{-j\omega}\right)^n u(n) \;\leftrightarrow\; \frac{1}{1-z^{-1}re^{-j\omega}} \, , \quad |z| > |r|$$

Using the linearity property, we obtain

$$X(z) = \frac{1}{1-z^{-1}re^{j\omega}} + \frac{1}{1-z^{-1}re^{-j\omega}} \, , \qquad |z| > |r|$$

Conjugation of a complex sequence:

$$\text{If} \quad X(z) = \mathcal{Z}\{x(n)\} \quad \text{then} \quad X^*(z^*) = \mathcal{Z}\{x^*(n)\}$$

$$\text{Also if} \quad x(n) = \mathcal{Z}^{-1}\{X(z)\} \; \text{then} \; x^*(n) = \mathcal{Z}^{-1}\{X^*(z^*)\}$$

$$\text{with} \;\; \mathrm{ROC} = \mathrm{ROC}_x$$

Time reversal:

$$\text{If} \quad X(z) = \mathcal{Z}\{x(n)\} \quad \text{then} \quad X(1/z) = \mathcal{Z}\{x(-n)\}$$

$$\text{Also if} \quad x(n) = \mathcal{Z}^{-1}\{X(z)\} \; \text{then} \; x(-n) = \mathcal{Z}^{-1}\{X(1/z)\}$$

$$\text{with} \;\; \mathrm{ROC} = 1/\mathrm{ROC}_x$$

Example: Consider the sequence

$$x(n) = a^{-n} u(-n)$$

which is a time-reversed version of

$$y(n) = a^n u(n) \quad \leftrightarrow \quad Y(z) = \frac{1}{1 - az^{-1}}\,, \qquad |z| > |a|$$

From the time reversal property

$$\begin{aligned} X(z) &= Y(1/z) \\ &= \frac{1}{1 - az}\,, \qquad |z| < |a^{-1}| \end{aligned}$$

Convolution of sequencies:

$$\text{If} \quad X(z) = \mathcal{Z}\{x(n)\} \quad \text{and} \quad Y(z) = \mathcal{Z}\{y(n)\}$$

$$\text{then} \quad X(z)Y(z) = \mathcal{Z}\{\{x(n)\} * \{y(n)\}\}$$

$$\text{Also if} \quad x(n) = \mathcal{Z}^{-1}\{X(z)\} \quad \text{and} \quad y(n) = \mathcal{Z}^{-1}\{Y(z)\}$$

$$\text{then} \quad \{x(n)\} * \{y(n)\} = \mathcal{Z}^{-1}\{X(z)Y(z)\}$$

$$\text{with} \;\; \mathrm{ROC} = \mathrm{ROC}_x \cap \mathrm{ROC}_y$$

Example: Evaluate the convolution of

$$x(n) = a^n u(n) \quad \text{and} \quad y(n) = u(n)$$

for $|a| < 1$. The z-transforms are

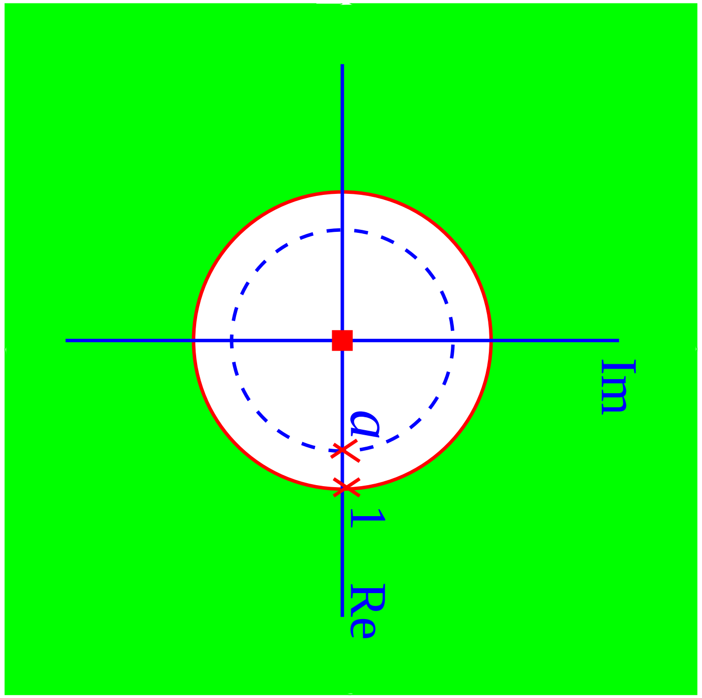
$$\begin{aligned} X(z) &= \frac{1}{1 - az^{-1}}, \quad |z| > a \\ Y(z) &= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \qquad \Longrightarrow \end{aligned}$$

$$\begin{aligned} \mathcal{Z}\{\{x(n)\} * \{y(n)\}\} &= X(z)Y(z) = \frac{z^2}{(z-a)(z-1)} \\ &= \frac{1}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right), \quad |z| > 1 \end{aligned}$$

Using the linearity property and the standard z-transform pairs, we obtain that

$$\{x(n)\} * \{y(n)\} = \frac{1}{1-a}\left(u(n) - a^{n+1}u(n)\right) \quad , \qquad |z| > 1$$

Pole-zero ROC plot and the results of convolution



4.6 Analysis of LTI Systems Using z-Transforms

From the convolution property

$$y(n) = \{h(n)\} * \{x(n)\} \quad \longleftrightarrow \quad Y(z) = H(z)X(z)$$

$$H(z) = \mathcal{Z}\{h(n)\} \quad \longleftarrow \quad \text{transfer function}$$

Interpretation of the transfer function

Result: A discrete-time LTI system is causal if and only if the ROC of its transfer function is the exterior of a circle including infinity.

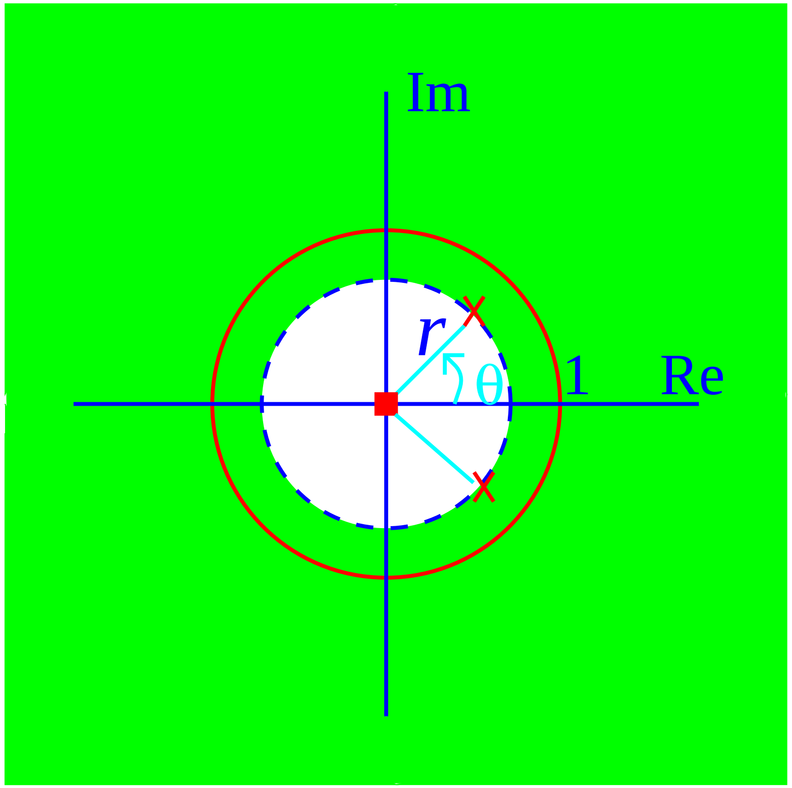
Proof: follows from the ROC properties and from the fact that $h(n)$ is a right-sided sequence.

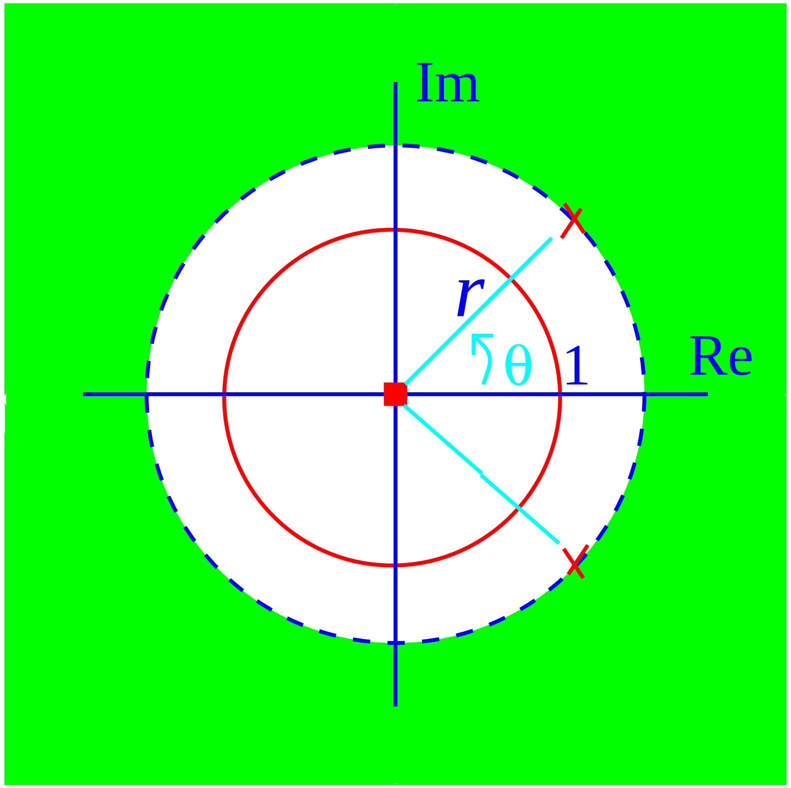
Result: A discrete-time LTI system is stable if and only if the ROC of its transfer function $H(z)$ includes the unit circle $|z| = 1$.

Proof: elementary, based on the definition of stability.

Example:

$$H(z) = \frac{1}{1 - [2r\cos\theta]z^{-1} + r^2 z^{-2}}$$





stable system ($r<1$) **unstable system ($r>1$)**

Recall on LCCD equations of ARMA processes

$$\sum_{k=0}^{N} a(k)y(n-k) = \sum_{k=0}^{M} b(k)x(n-k) \qquad (*)$$

Taking z-transforms of both sides of $(*)$

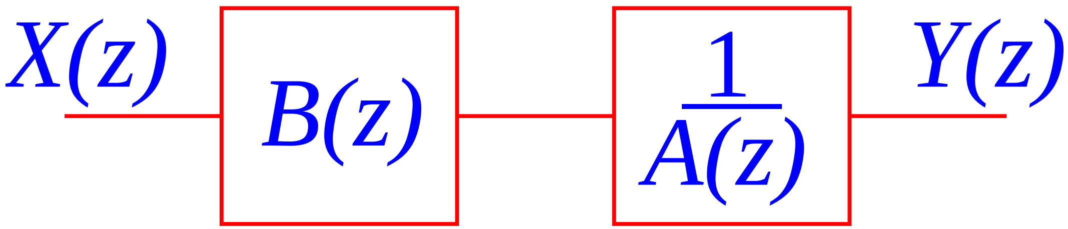
$$\sum_{k=0}^{N} a(k)\mathcal{Z}\{y(n-k)\} = \sum_{k=0}^{M} b(k)\mathcal{Z}\{x(n-k)\}$$

and using time-shifting property, we obtain

$$Y(z)\sum_{k=0}^{N} a(k)z^{-k} = X(z)\sum_{k=0}^{M} b(k)z^{-k}$$

Hence, the transfer function of an ARMA process

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b(k) z^{-k}}{\sum_{k=0}^{N} a(k) z^{-k}} = \frac{B(z)}{A(z)}$$



5 DISCRETE AND FAST FOURIER TRANSFORMS

5.1 Discrete Fourier Transform (DFT)

Recall the DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

The DTFT is not suitable for a practical DSP because:

- in any DSP application, we are able to compute the spectrum only at specific discrete values of ω
- any signal in any DSP application can be measured only in a finite number of points

A finite signal measured at N points:

$$x(n) = \begin{cases} 0\,, & n < 0 \\ y(n)\,, & 0 \le n \le (N-1) \\ 0\,, & n \ge N \end{cases}$$

where $y(n)$ are the measurements taken in N points.

Let us sample the spectrum $X(\omega)$ in frequency domain so that

$$X(k) = X(k\,\Delta\omega)\,, \quad \Delta\omega = \frac{2\pi}{N} \qquad \Longrightarrow$$

$$X(k) \;=\; \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi\frac{kn}{N}} \qquad \text{DFT}$$

Result: the inverse DFT is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi \frac{kn}{N}}$$

Proof:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left\{ \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi \frac{km}{N}} \right\} e^{j2\pi \frac{kn}{N}}$$

$$= \sum_{m=0}^{N-1} x(m) \underbrace{\left\{ \frac{1}{N} \sum_{k=0}^{N-1} e^{-j2\pi \frac{k(m-n)}{N}} \right\}}_{\delta(m-n)} = x(n)$$

The DFT transform:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi\frac{kn}{N}} \qquad \text{analysis}$$

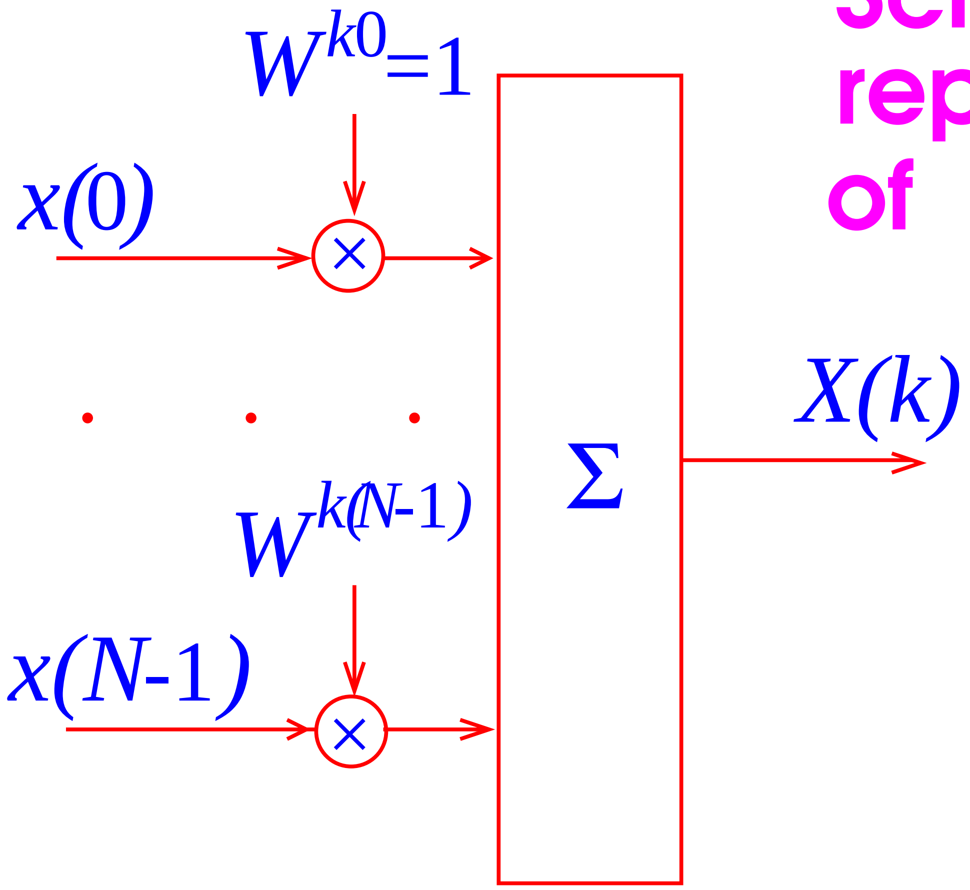
$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi\frac{kn}{N}} \qquad \text{synthesis}$$

Alternative formulation:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W^{kn} \qquad \longleftarrow \qquad W = e^{-j\frac{2\pi}{N}}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W^{-kn}$$

Schematic representation of DFT



An important property of DFT-spectrum:

$$
\begin{aligned}
X(k+N) &= \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi\frac{(k+N)n}{N}} \\
&= \left(\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi\frac{kn}{N}}\right) e^{-j2\pi n} \\
&= X(k)\, e^{-j2\pi n} = X(k) \qquad \Longrightarrow
\end{aligned}
$$

the DFT-spectrum $X(k)$ is periodic with period N (recall that DTFT-spectrum is periodic as well, but with period 2π)

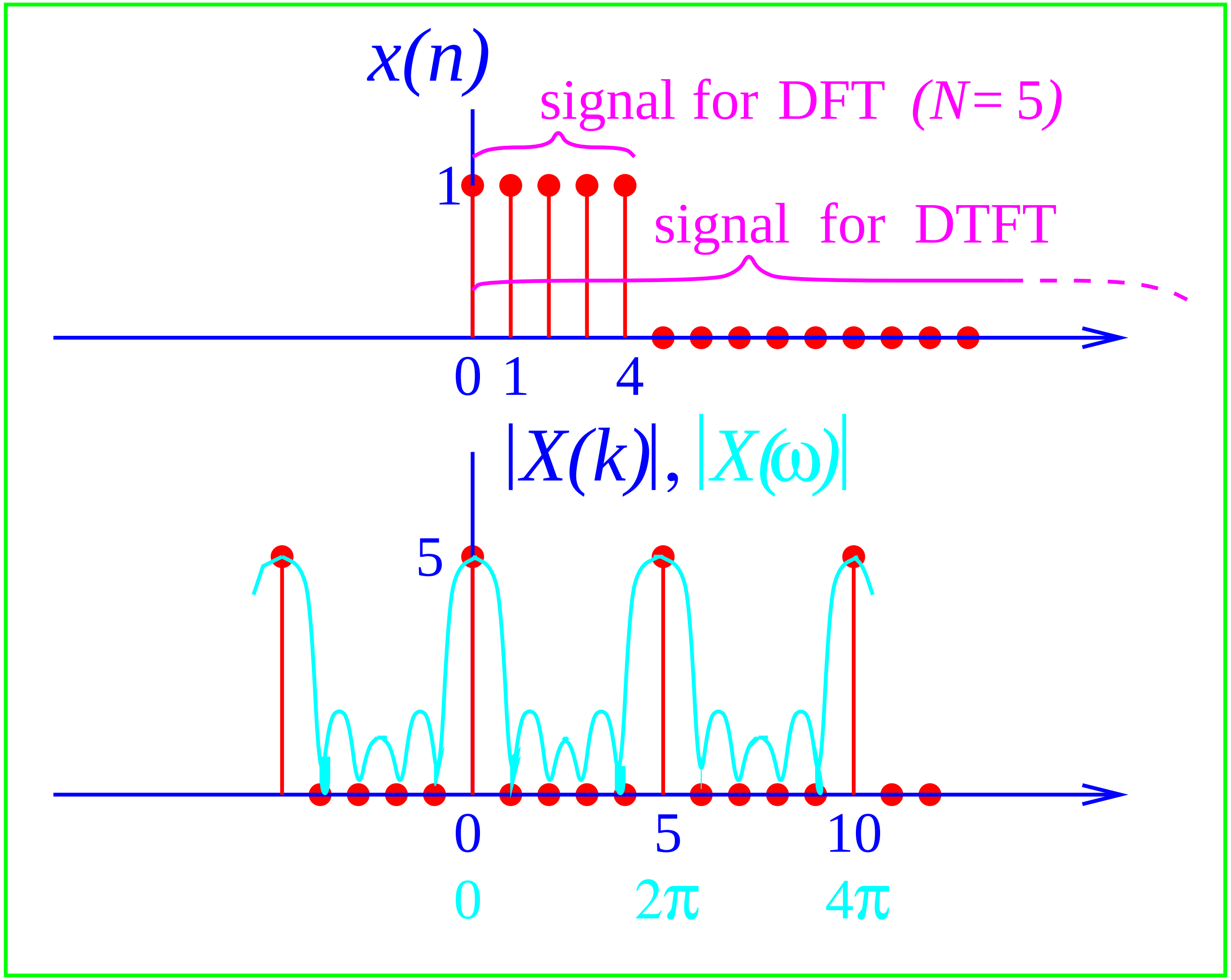
Example: DFT of rectangular pulse

$$x(n) = \begin{cases} 1\,, & 0 \le n \le (N-1) \\ 0\,, & \text{otherwise} \end{cases}$$

$$X(k) = \sum_{n=0}^{N-1} e^{-j2\pi\frac{kn}{N}} = N\,\delta(k) \qquad \Longrightarrow$$

the rectangular pulse is "interpreted" by the DFT as a spectral line at the frequency $\omega = 0$

DFT and DTFT of a rectangular pulse ($N = 5$)



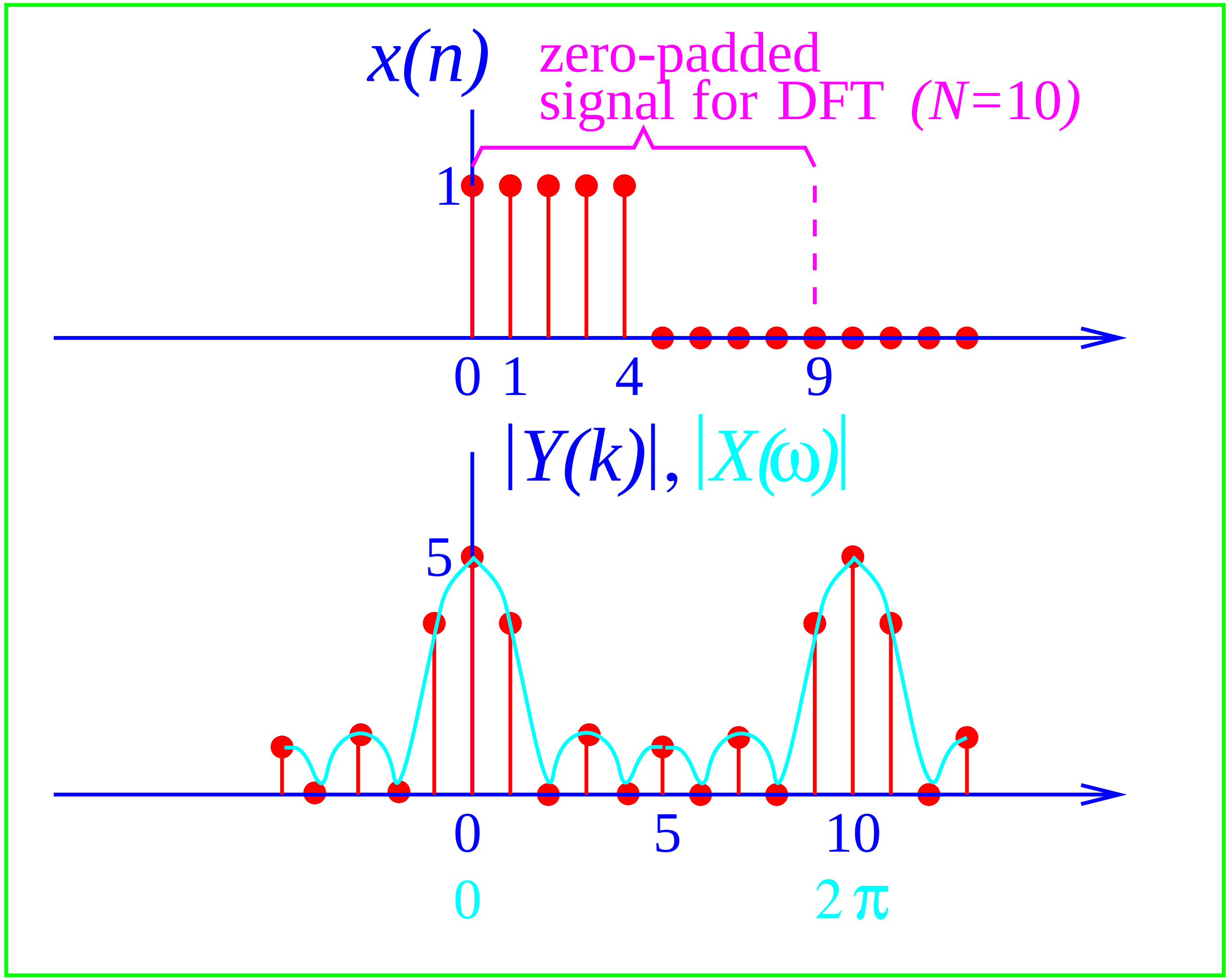
Example (continued): What happens with the DFT of this rectangular pulse if we'll increase N, i.e., include in the number of signal points several zeros? That is, let the DFT sequence be

$$\{y(n)\} = \{x(0), \ldots, x(M-1), \underbrace{0, 0, \ldots, 0}_{N-M \text{ positions}}\}$$

where $x(0) = \cdots = x(M-1) = 1$. Hence, the DFT:

$$\begin{aligned} Y(k) &= \sum_{n=0}^{N-1} y(n)\, e^{-j2\pi\frac{kn}{N}} = \sum_{n=0}^{M-1} e^{-j2\pi\frac{kn}{N}} \\ &= \frac{\sin\left(\pi\frac{kM}{N}\right)}{\sin\left(\pi\frac{k}{N}\right)}\, e^{-j\pi\frac{k(M-1)}{N}} \end{aligned}$$

DFT and DTFT of a rectangular pulse with zero-padding ($N = 10$, $M = 5$)



Corollary: using more and more zero-padding on analyzed sequence, we are able to "approximate" its DTFT better and better

Corollary: zero-padding cannot improve the resolution of spectral components because the resolution is proportional to $1/M$ rather than $1/N$

Remark: zero-padding will be a very important tool for a fast implementation of DFT

5.2 Matrix Formulation of DFT

Introduce the $N \times 1$ vectors

$$\begin{aligned} \mathbf{x} &= [x(0), x(1), \ldots, x(N-1)]^T \\ \mathbf{X} &= [X(0), X(1), \ldots, X(N-1)]^T \end{aligned}$$

and the $N \times N$ matrix

$$\mathbf{W} = \begin{bmatrix} W^0 & W^0 & W^0 & \ldots & W^0 \\ W^0 & W^1 & W^2 & \ldots & W^{N-1} \\ W^0 & W^2 & W^4 & \ldots & W^{2(N-1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ W^0 & W^{N-1} & W^{2(N-1)} & \ldots & W^{(N-1)^2} \end{bmatrix}$$

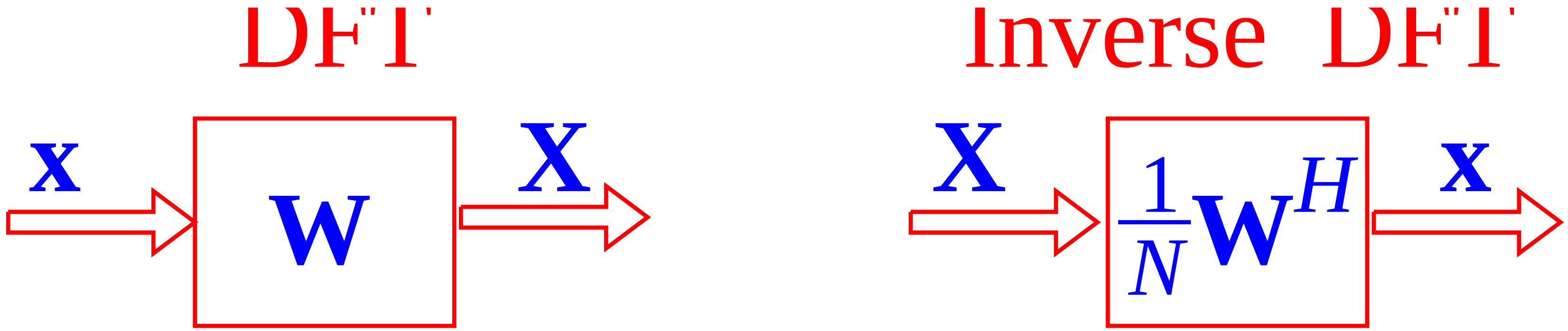
DFT in a matrix form:

$$\mathbf{X} = \mathbf{W}\mathbf{x}$$

Result: The inverse DFT is given by

$$\mathbf{x} = \frac{1}{N}\mathbf{W}^H\mathbf{X}$$

Proof: elementary, using the direct checking of the fact that $\mathbf{W}^H\mathbf{W} = \mathbf{W}\mathbf{W}^H = N\,\mathbf{I}$, where $\mathbf{I}$ is the identity matrix.



5.3 Frequency Interval and Frequency Resolution

Since the DFT is periodic with the period N, its frequency resolution

$$\Delta f \simeq \frac{1}{N\,\Delta t} \quad [\mathrm{Hz}]$$

and covered frequency interval

$$\Delta F \simeq N\,\Delta f = \frac{1}{\Delta t} \quad [\mathrm{Hz}] \qquad \Longrightarrow$$

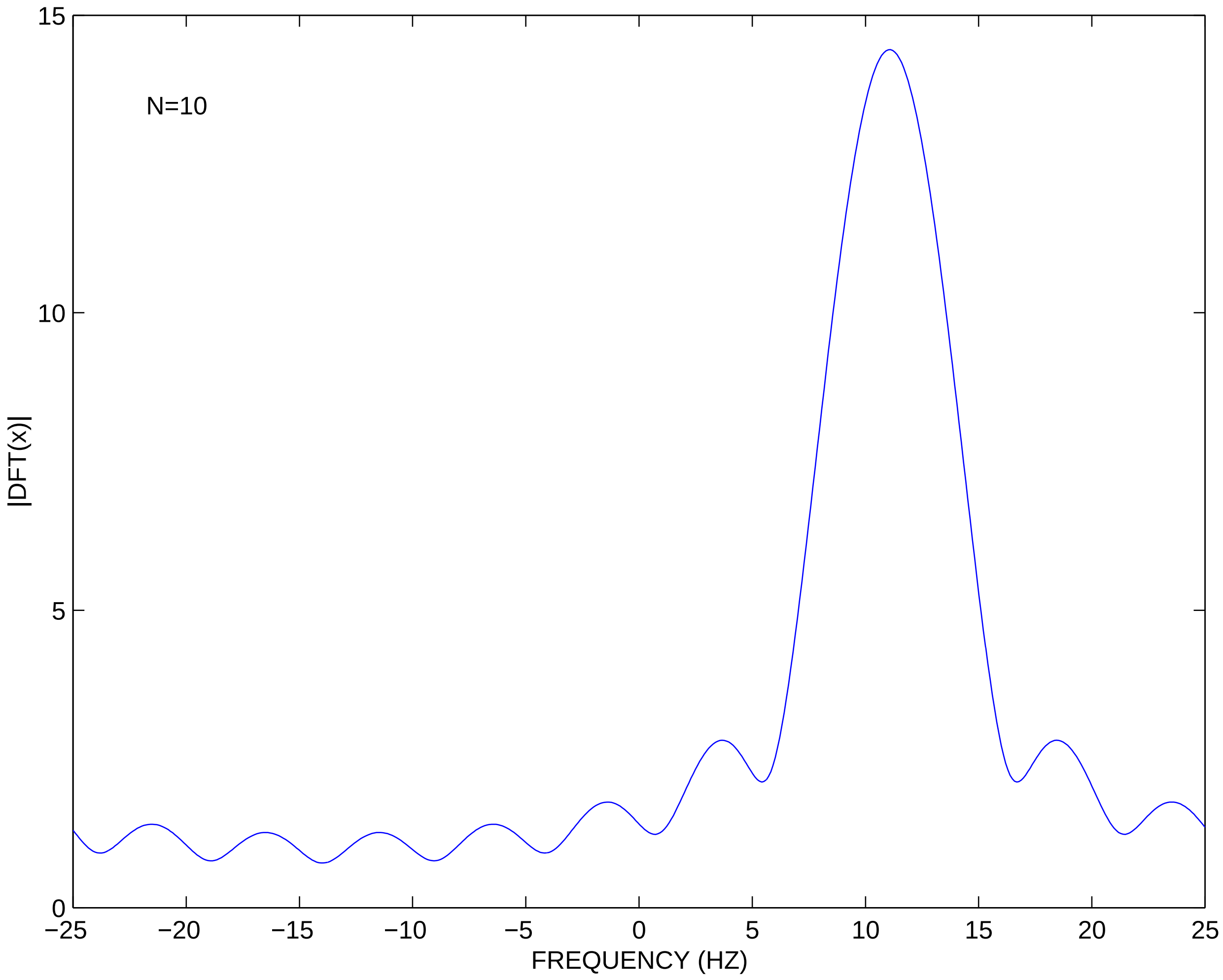
the frequency resolution is determined only by the inversed length of the observation interval, whereas the frequency interval is determined only by the inversed length of sampling interval $\Longrightarrow$ increasing the sampling rate we can expand the frequency interval, and increasing the observation time, we can improve frequency resolution!

Question: Does zero-padding alter the frequency resolution?

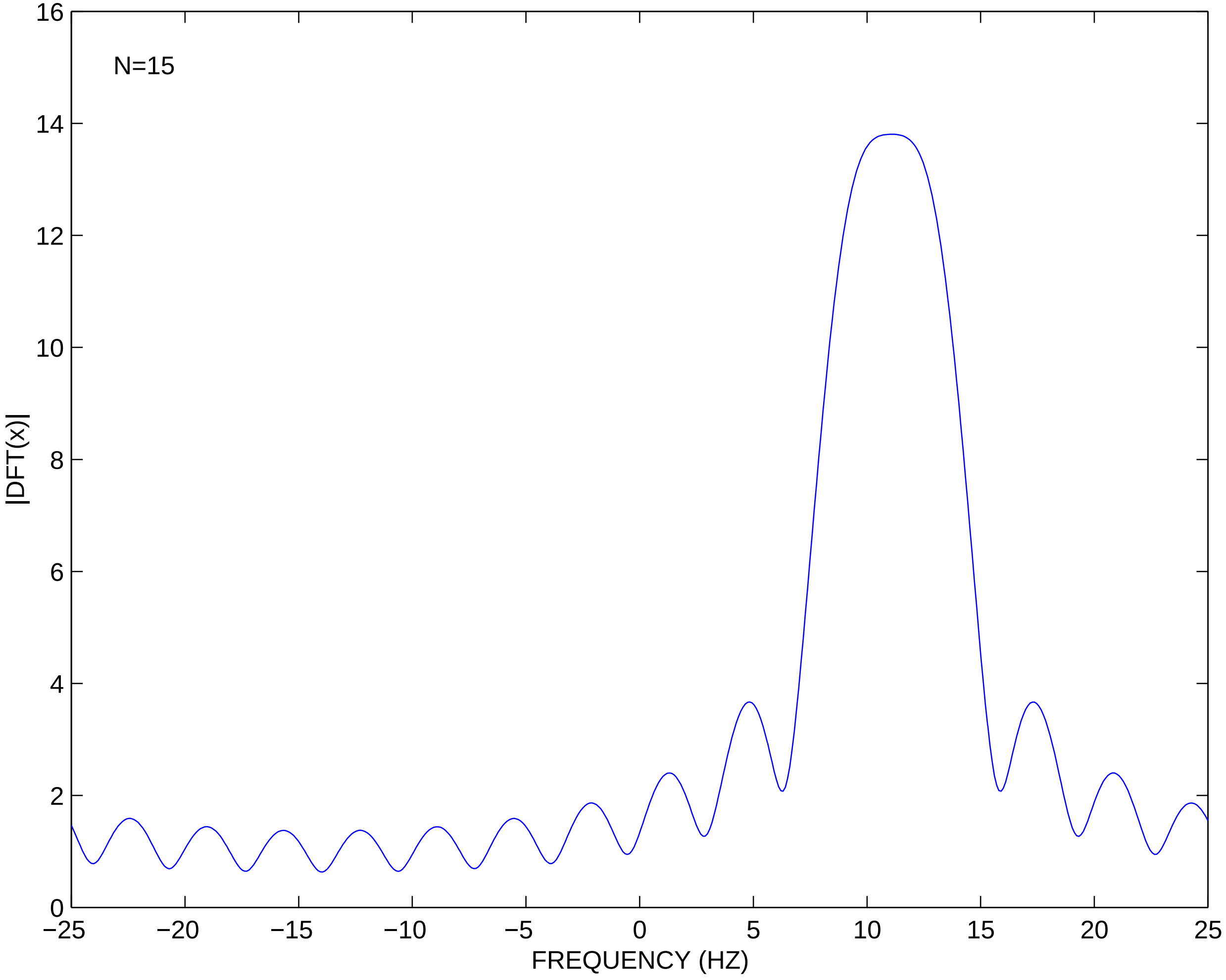
The answer is negative because the resolution is determined only by the length of observation interval, and, obviously, zero-padding does not increase this length

Example: Let two complex exponentials with the close frequencies $f_1 = 10$ Hz and $f_2 = 12$ Hz be sampled with the sampling interval $\Delta t = 0.02$ seconds and let us consider various data lengths $N = 10, 15, 30, 100, 300$ with zero-padding of each data to 512 points.

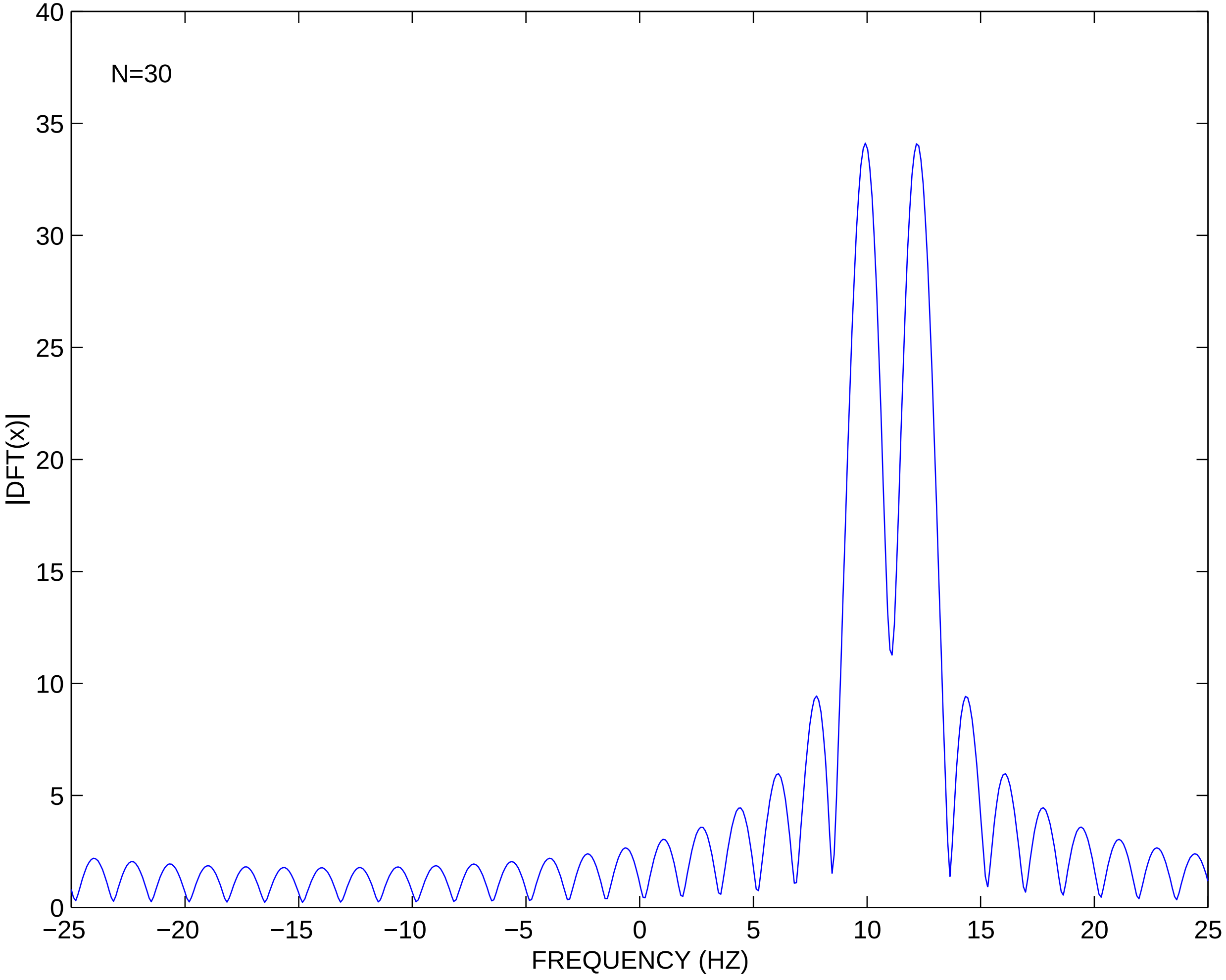
DFT with $N = 10$ and zero-padding to 512 points. The signals are unresolved because $f_2 - f_1 = 2$ Hz $< 1/(N\Delta t) = 5$ Hz



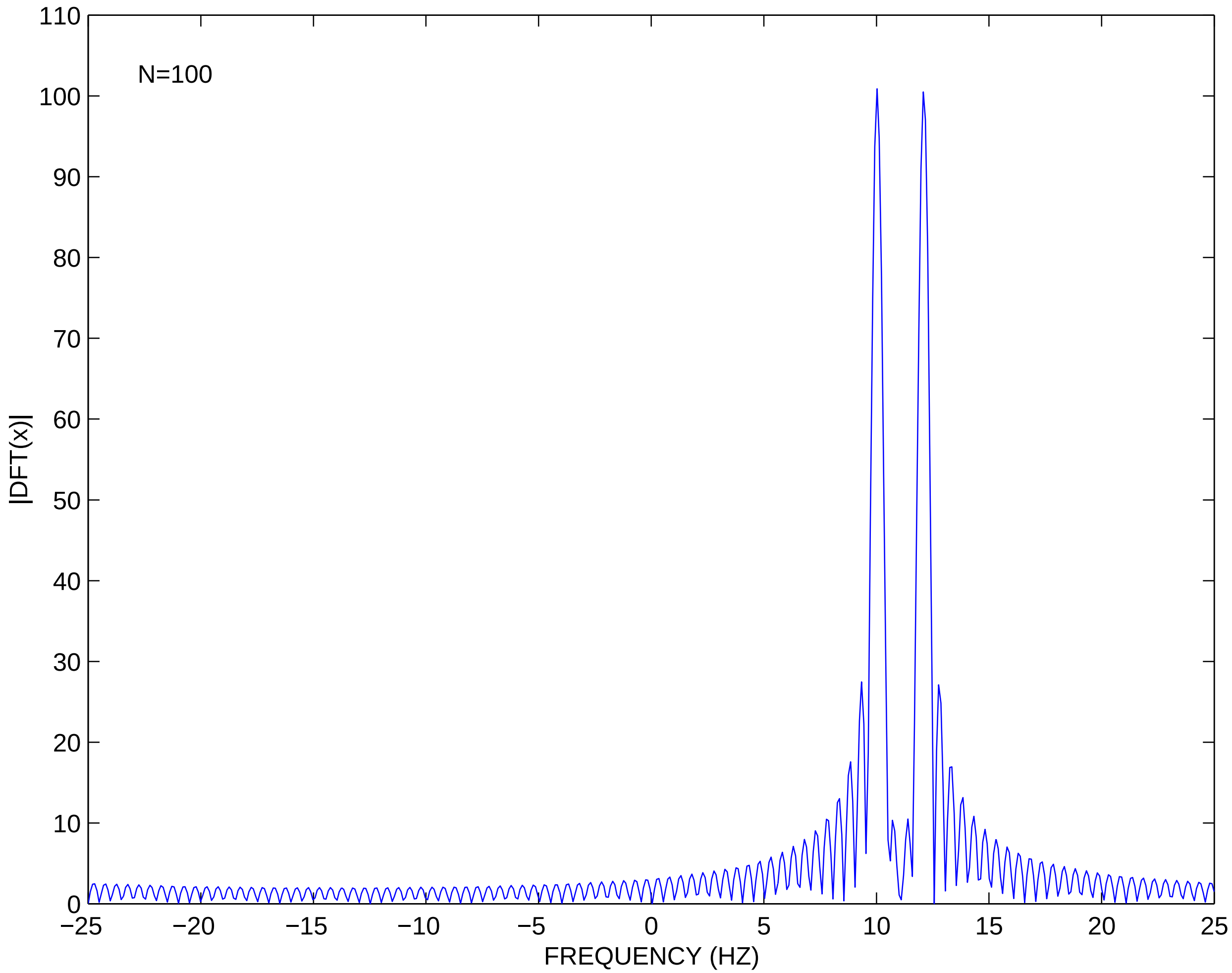
DFT with $N = 15$ and zero-padding to 512 points. The signals are still unresolved because $f_2 - f_1 = 2 \text{ Hz} < 1/(N\Delta t) \simeq 3.3 \text{ Hz}$



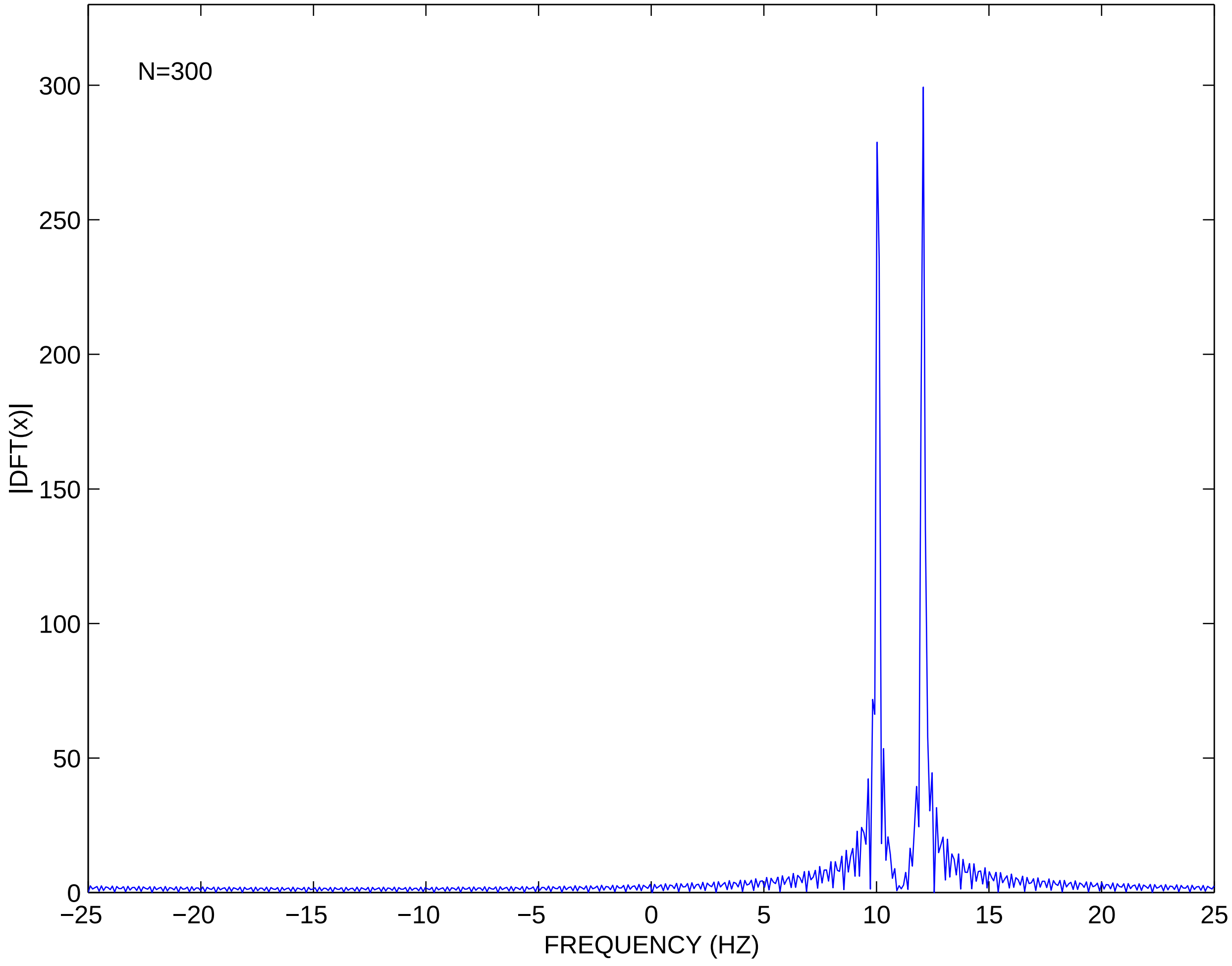
DFT with $N = 30$ and zero-padding to 512 points. The signals are resolved because $f_2 - f_1 = 2\ \mathrm{Hz} > 1/(N\Delta t) \simeq 1.7\ \mathrm{Hz}$



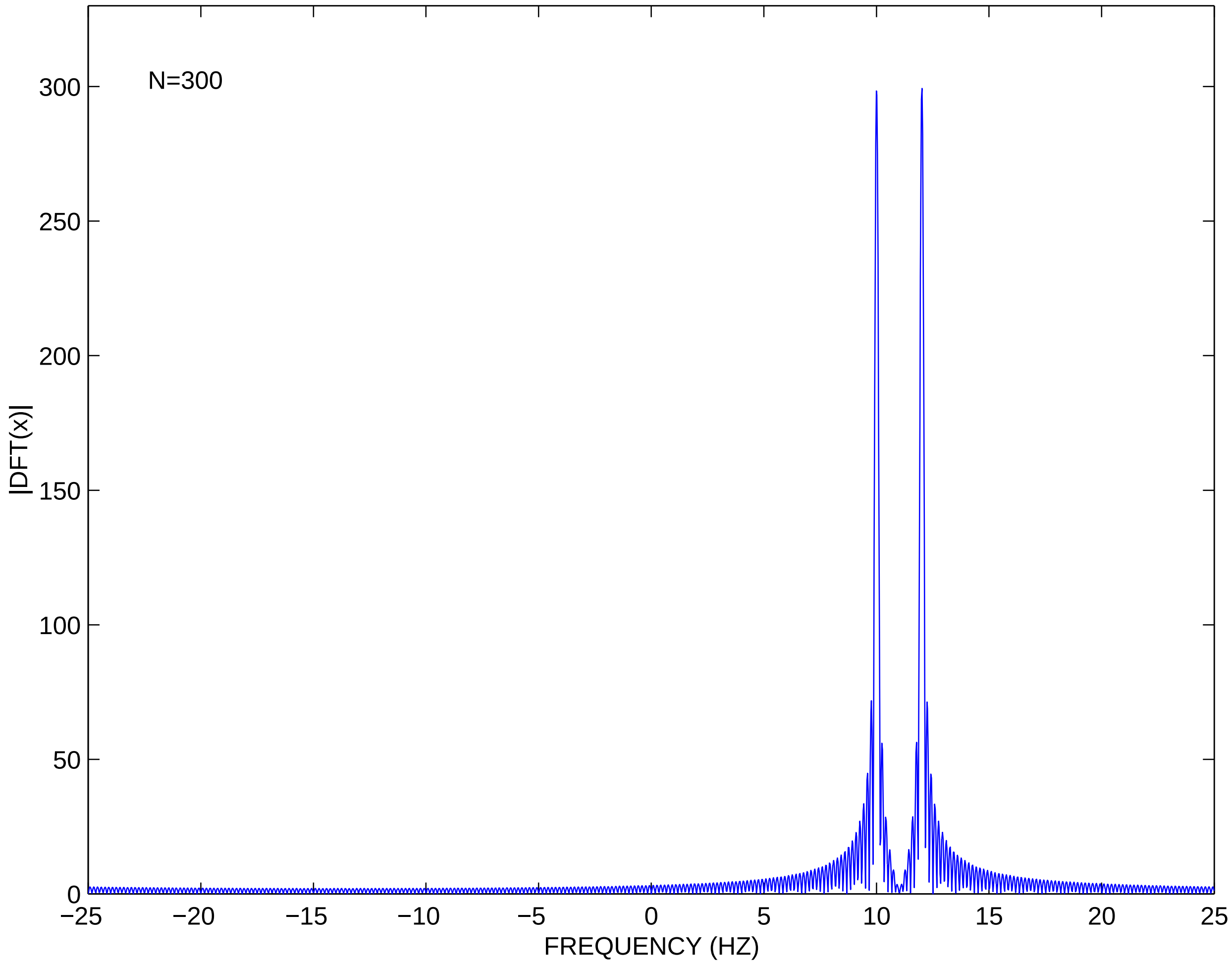
DFT with $N = 100$ and zero-padding to 512 points. The signals are well resolved because $f_2 - f_1 = 2\ \mathrm{Hz} > 1/(N\Delta t) = 0.5\ \mathrm{Hz}$



DFT with $N = 300$ and zero-padding to 512 points. The signals are fine resolved because $f_2 - f_1 = 2\ \mathrm{Hz} \gg 1/(N\Delta t) \simeq 0.17\ \mathrm{Hz}$



Better representation of the previous DFT plot by means of increasing the number of zeros: zero-padding to 2048 points.



5.4 Interpretation of DFT via Discrete Fourier Series

Construct a periodic sequence by "repeating" the finite sequence $x(n)$, $n = 0, \dots, N-1$:

$$\{\tilde{x}(n)\} = \{\dots, \underbrace{x(0), \dots, x(N-1)}_{\{x(n)\}}, \underbrace{x(0), \dots, x(N-1)}_{\{x(n)\}}, \dots\}$$

The discrete version of the Fourier Series can be written as

$$\begin{aligned} \tilde{x}(n) &= \sum_k X_k \, e^{j2\pi \frac{kn}{N}} \\ &= \frac{1}{N} \sum_k \tilde{X}(k) \, e^{j2\pi \frac{kn}{N}} \end{aligned}$$

where $\tilde{X}(k) = N X_k$

Remarking that

$$W^{-kn} = e^{j2\pi\frac{kn}{N}} = e^{j2\pi\frac{(k+mN)n}{N}} = W^{-(k+mN)n}$$

for integer values of m, we obtain that the summation in the Discrete Fourier Series (DFS) should contain only N terms:

$$\tilde{x}(n) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(k)\, e^{j2\pi\frac{kn}{N}} \qquad \text{DFS}$$

The DFS coefficients are given by

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j2\pi\frac{kn}{N}} \qquad \text{inverse DFS}$$

Proof:

$$\sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j2\pi\frac{kn}{N}} = \sum_{n=0}^{N-1} \left\{ \frac{1}{N} \sum_{p=0}^{N-1} \tilde{X}(p)\, e^{j2\pi\frac{pn}{N}} \right\} e^{-j2\pi\frac{kn}{N}}$$

$$= \sum_{p=0}^{N-1} \tilde{X}(p) \underbrace{\left\{ \frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi\frac{(p-k)n}{N}} \right\}}_{\delta(p-k)} = \tilde{X}(k)$$

The DFS pair:

$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n)\, e^{-j2\pi\frac{kn}{N}} \qquad \text{analysis}$$

$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k)\, e^{j2\pi\frac{kn}{N}} \qquad \text{synthesis}$$

Remarks:

- The DFS and DFT pairs are identical except for the fact that the DFT is applied to a finite (nonperiodic) sequence $x(n)$, whereas the DFS is applied to a periodic sequence $\tilde{x}(n)$
- The conventional (continuous-time) FS represent a periodic signal using an infinite number of complex exponentials, whereas the DFS represent such a signal using a finite number of complex exponentials

5.5 Properties of the DFT

Linearity:

$$\text{If} \quad X(k) = \mathcal{DFT}\{x(n)\} \quad \text{and} \quad Y(k) = \mathcal{DFT}\{y(n)\}$$

$$\text{then} \quad a\,X(k) + b\,Y(k) = a\,\mathcal{DFT}\{x(n)\} + b\,\mathcal{DFT}\{y(n)\}$$

where the lengths of both sequences should be equalized by means of zero-padding.

$$\text{Also if} \quad x(n) = \mathcal{DFT}^{-1}\{X(k)\} \quad \text{and} \quad y(n) = \mathcal{DFT}^{-1}\{Y(k)\}$$

$$\text{then} \quad a\,x(n) + b\,y(n) = a\,\mathcal{DFT}^{-1}\{X(k)\} + b\,\mathcal{DFT}^{-1}\{Y(k)\}$$

Circular shift of a sequence:

$$\text{If} \quad X(k) = \mathcal{DFT}\{x(n)\}$$

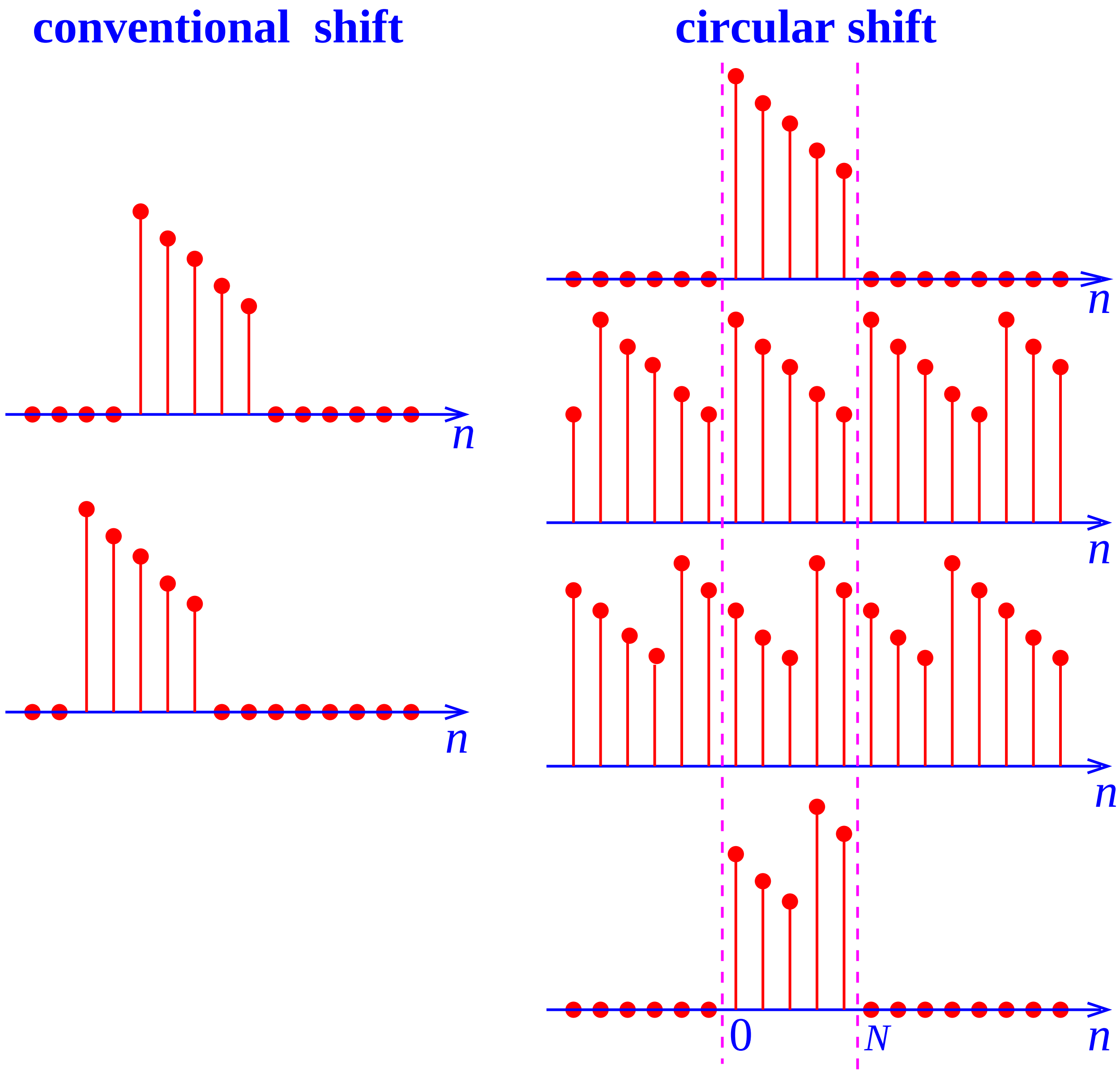
$$\text{then} \quad X(k)e^{-j2\pi\frac{km}{N}} = \mathcal{DFT}\{x\left((n-m)\text{mod}\, N\right)\}$$

$$\text{Also if} \quad x(n) = \mathcal{DFT}^{-1}\{X(k)\} \;\; \text{then}$$

$$x\left((n-m)\text{mod}\, N\right) = \mathcal{DFT}^{-1}\{X(k)e^{-j2\pi\frac{km}{N}}\}$$

where the operation $\mod N$ is exploited for denoting the periodic extension $\tilde{x}(n)$ of the signal $x(n)$:

$$\tilde{x}(n) = x(n \mod N)$$


conventional shift
circular shift
n
n
n
n
n
0
N
n

Proof of the circular shift property:

$$
\begin{aligned}
\sum_{n=0}^{N-1} x((n-m) \bmod N)\, W^{kn} &= \sum_{n=0}^{N-1} x\left((n-m)\mathrm{mod}N\right) W^{k(n-m+m)} \\
&= W^{km} \sum_{n=0}^{N-1} x((n-m)\mathrm{mod}N) W^{k(n-m)} \\
&= W^{km} \sum_{n=0}^{N-1} x((n-m)\mathrm{mod}N) \\
&\cdot\; W^{k(n-m) \bmod N} = W^{km} X(k)
\end{aligned}
$$

where we use the facts that $W^{k(l \bmod N)} = W^{kl}$ and that the order of summation in DFT does not change its result

Frequency shift (modulation):

$$\text{If} \quad X(k) = \mathcal{DFT}\{x(n)\}$$

$$\text{then} \quad X((k-m)\text{mod}\, N) = \mathcal{DFT}\{x(n)e^{j2\pi\frac{mn}{N}}\}$$

$$\text{Also if} \quad x(n) = \mathcal{DFT}^{-1}\{X(k)\} \ \text{then}$$

$$x(n)e^{j2\pi\frac{mn}{N}} = \mathcal{DFT}^{-1}\{X((k-m)\text{mod}\, N)\}$$

Proof: similar to that for the circular shift property

Parseval theorem:

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k) \qquad \text{general form}$$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2 \qquad \text{specific form}$$

Proof: Using the matrix formulation of the DFT, we obtain

$$\mathbf{y}^H\mathbf{x} = \left(\frac{1}{N}\mathbf{W}^H\mathbf{Y}\right)^H \left(\frac{1}{N}\mathbf{W}^H\mathbf{X}\right)$$

$$= \frac{1}{N^2}\mathbf{Y}^H \underbrace{\mathbf{W}\mathbf{W}^H}_{N\mathbf{I}}\mathbf{X} = \frac{1}{N}\mathbf{Y}^H\mathbf{X}$$

Conjugation:

If $X(k) = \mathcal{DFT}\{x(n)\}$ then $X^*((N-k)\text{mod}\, N) = \mathcal{DFT}\{x^*(n)\}$

Also if $x(n) = \mathcal{DFT}^{-1}\{X(k)\}$ then

$$x^*(n) = \mathcal{DFT}^{-1}\{X^*((N-k)\text{mod}\, N)\}$$

Proof:

$$\sum_{n=0}^{N-1} x^*(n) W^{kn} = \left[\sum_{n=0}^{N-1} x(n) W^{-kn}\right]^* = \left[\sum_{n=0}^{N-1} x(n)\, \underbrace{W^{(-k \bmod N)n}}_{W^{-kn}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n) W^{[(N-k)\text{mod}\, N]n}\right]^* = X^*((N-k)\text{mod} N)$$

Circular convolution:

$$\text{If} \quad X(k) = \mathcal{DFT}\{x(n)\} \quad \text{and} \quad Y(k) = \mathcal{DFT}\{y(n)\}$$

$$\text{then} \quad X(k)Y(k) = \mathcal{DFT}\{\{x(n)\} \circledast \{y(n)\}\}$$

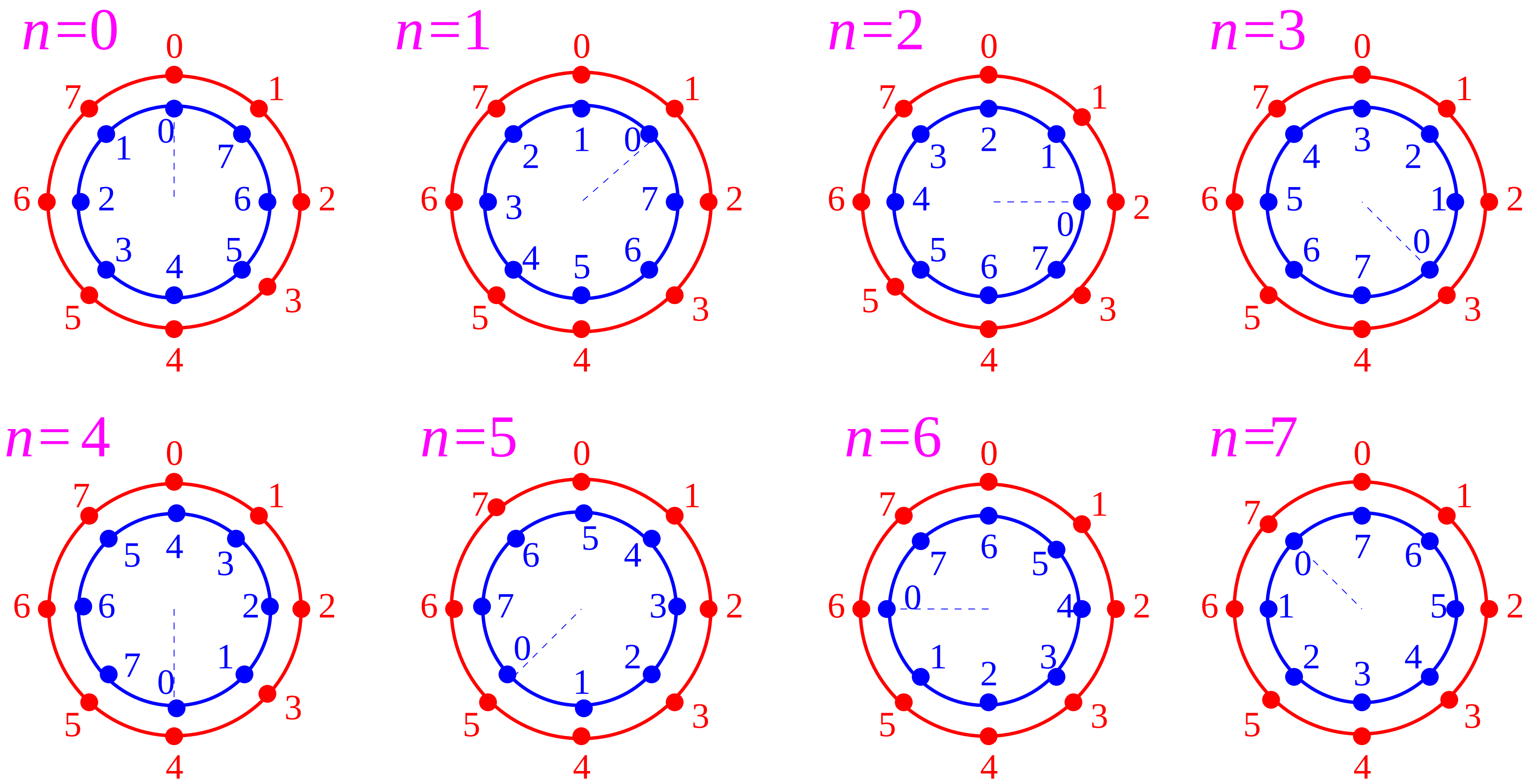
$$\text{Also if} \quad x(n) = \mathcal{DFT}^{-1}\{X(k)\} \quad \text{and} \quad y(n) = \mathcal{DFT}^{-1}\{y(k)\}$$

$$\text{then} \quad x(n) \circledast y(n) = \mathcal{DFT}^{-1}\{X(k)Y(k)\}$$

Here $\circledast$ stands for circular convolution defined by

$$\{x(n)\} \circledast \{y(n)\} = \sum_{m=0}^{N-1} x(m) y((n-m) \bmod N)$$

Illustration of circular convolution for $N = 8$:



• - *x(n)* spread clockwise

• - *y(n)* spread counterclockwise

Example: Let the circularily convolved sequencies are

$$\begin{aligned}\{x(n)\} &= \{1, -1, -1, -1, 1, 0, 1, 2\} \\ \{y(n)\} &= \{5, -4, 3, 2, -1, 1, 0, -1\}\end{aligned}$$

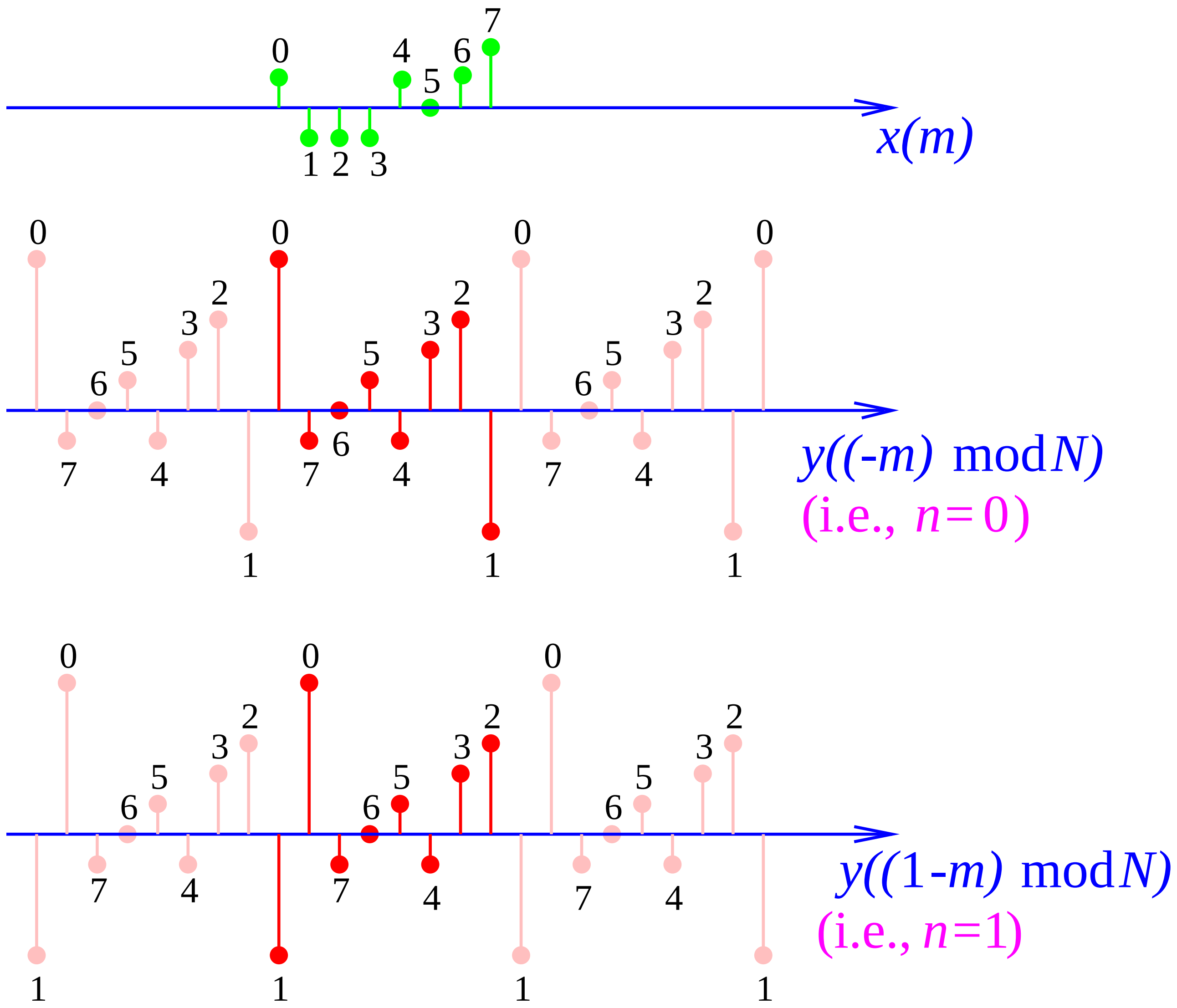
and $\{z(n)\} = \{x(n)\} \circledast \{y(n)\}$. Then

$$\begin{aligned}z(0) &= x(0)y(0) + x(1)y(7) + x(2)y(6) + x(3)y(5) + x(4)y(4) \\ &+ x(5)y(3) + x(6)y(2) + x(7)y(1) = -1 \\ z(1) &= x(0)y(1) + x(1)y(0) + x(2)y(7) + x(3)y(6) + x(4)y(5) \\ &+ x(5)y(4) + x(6)y(3) + x(7)y(2) = 1 \\ z(2) &= x(0)y(2) + x(1)y(1) + x(2)y(0) + x(3)y(7) + x(4)y(6) \\ &+ x(5)y(5) + x(6)y(4) + x(7)y(3) = 6\end{aligned}$$

Example (continued):

$$
\begin{aligned}
z(3) &= x(0)y(3) + x(1)y(2) + x(2)y(1) + x(3)y(0) + x(4)y(7) \\
&+ x(5)y(6) + x(6)y(5) + x(7)y(4) = -4 \\
z(4) &= x(0)y(4) + x(1)y(3) + x(2)y(2) + x(3)y(1) + x(4)y(0) \\
&+ x(5)y(7) + x(6)y(6) + x(7)y(5) = 5 \\
z(5) &= x(0)y(5) + x(1)y(4) + x(2)y(3) + x(3)y(2) + x(4)y(1) \\
&+ x(5)y(0) + x(6)y(7) + x(7)y(6) = -8 \\
z(6) &= x(0)y(6) + x(1)y(5) + x(2)y(4) + x(3)y(3) + x(4)y(2) \\
&+ x(5)y(1) + x(6)y(0) + x(7)y(7) = 4 \\
z(7) &= x(0)y(7) + x(1)y(6) + x(2)y(5) + x(3)y(4) + x(4)y(3) \\
&+ x(5)y(2) + x(6)y(1) + x(7)y(0) = 7
\end{aligned}
$$

Example (continued): illustration of the circular convolution process



Proof of circular convolution property:

$$\mathcal{DFT}\{\{x(n)\}\circledast\{y(n)\}\} = \sum_{n=0}^{N-1} \underbrace{\left[\sum_{m=0}^{N-1} x(m)y((n-m)\bmod N)\right]}_{\{x(n)\}\circledast\{y(n)\}} W^{kn}$$

$$= \sum_{m=0}^{N-1} \underbrace{\left[\sum_{n=0}^{N-1} y((n-m)\bmod N)W^{kn}\right]}_{Y(k)W^{km}} x(m)$$

$$= Y(k) \underbrace{\sum_{m=0}^{N-1} x(m)W^{km}}_{X(k)} = X(k)Y(k)$$

Multiplication:

$$\text{If} \quad X(k) = \mathcal{DFT}\{x(n)\} \quad \text{and} \quad Y(k) = \mathcal{DFT}\{y(n)\}$$

$$\text{then} \quad X(k) \circledast Y(k) = \mathcal{DFT}\{x(n)y(n)\}$$

$$\text{Also if} \quad x(n) = \mathcal{DFT}^{-1}\{X(k)\} \quad \text{and} \quad y(n) = \mathcal{DFT}^{-1}\{y(k)\}$$

$$\text{then} \quad x(n)y(n) = \mathcal{DFT}^{-1}\{X(k) \circledast Y(k)\}$$

Proof: similar to that for the circular convolution property

5.6 The Fast Fourier Transform

Let us now introduce the following new notation

$$X_N(k) = X(k)\,,\ k = 0, \ldots, N-1\,, \qquad W_N = e^{-j\frac{2\pi}{N}}$$

Before, we used simpler notation $X(k)$ and W (without the subscript $_N$) because the sequence length was not important. Now, it is important!

Assume that

$$N = 2^r\,, \quad r = \log_2 N$$

i.e., that the transformed sequence length is an integer power of 2.

If the original sequence length does not satisfy this assumption, it can be always padded with a proper number of zeros so that the length of zero-padded sequence will satisfy this assumption!

The idea of the radix-2 FFT:

$$X_N(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad \longleftarrow \quad N\text{–point DFT}$$

$$= \sum_{n \text{ even}} x(n) W_N^{kn} + \sum_{n \text{ odd}} x(n) W_N^{kn}$$

$$= \sum_{l=0}^{N/2-1} x(2l) W_N^{2lk} + \sum_{l=0}^{N/2-1} x(2l+1) W_N^{(2l+1)k}$$

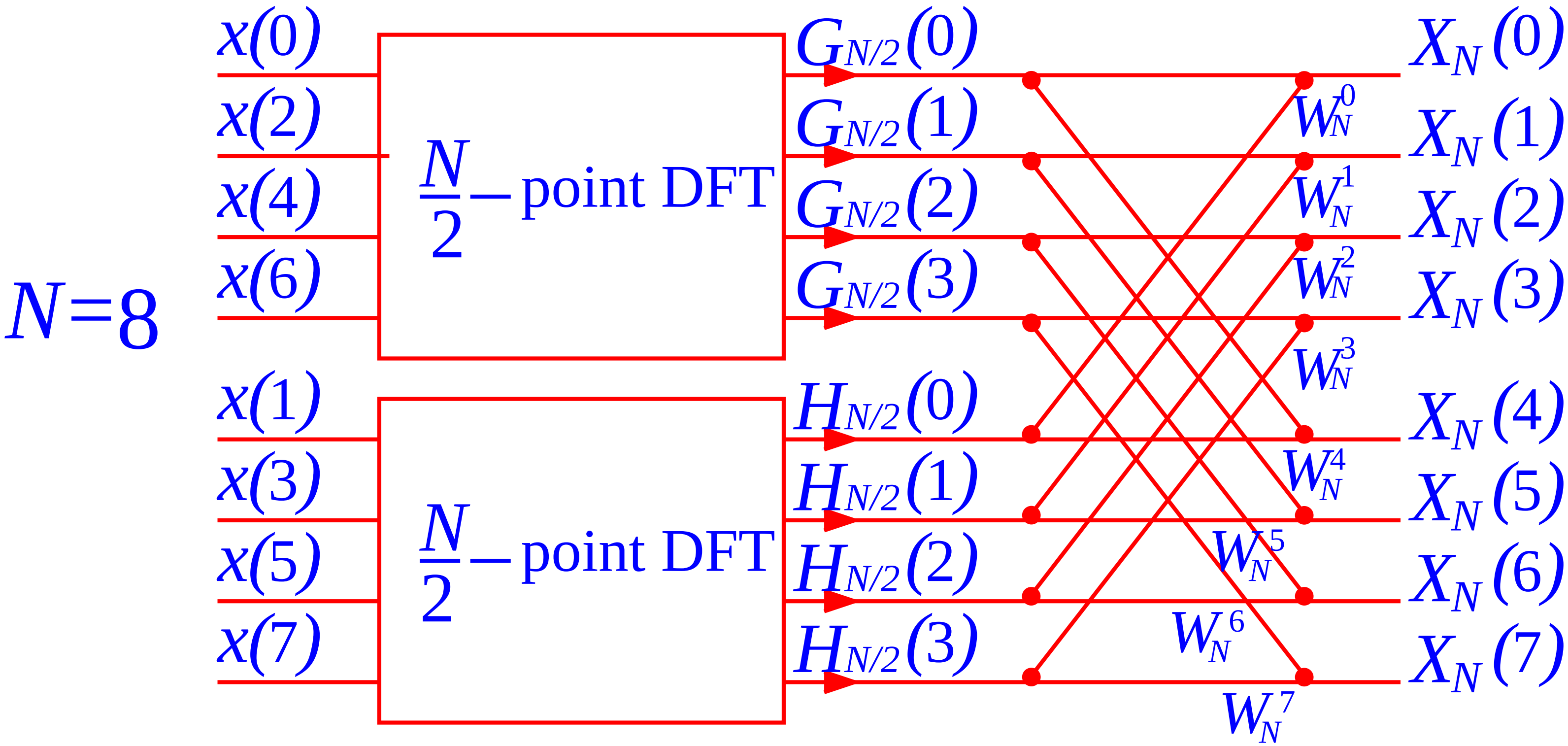
$$= \sum_{l=0}^{N/2-1} x(2l) (\underbrace{W_N^2}_{W_{N/2}})^{lk} + W_N^k \sum_{l=0}^{N/2-1} x(2l+1) (\underbrace{W_N^2}_{W_{N/2}})^{lk}$$

$$= \sum_{l=0}^{N/2-1} x(2l) W_{N/2}^{lk} + W_N^k \sum_{l=0}^{N/2-1} x(2l+1) W_{N/2}^{lk} \qquad \longleftarrow \quad \text{two } N/2\text{–point DFT's!!!}$$

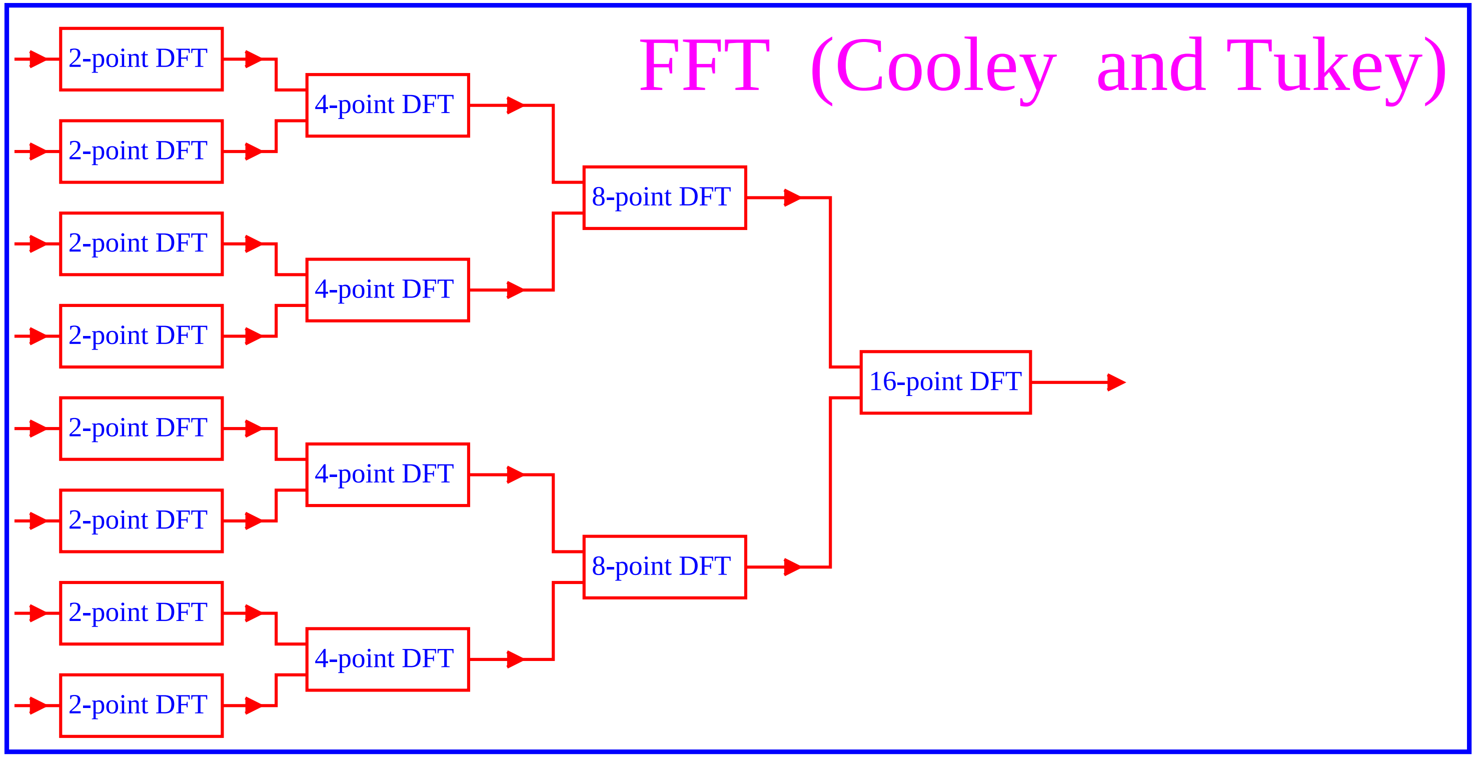
In a short notation:

$$X_N(k) = G_{N/2}(k) + W_N^k H_{N/2}(k)$$

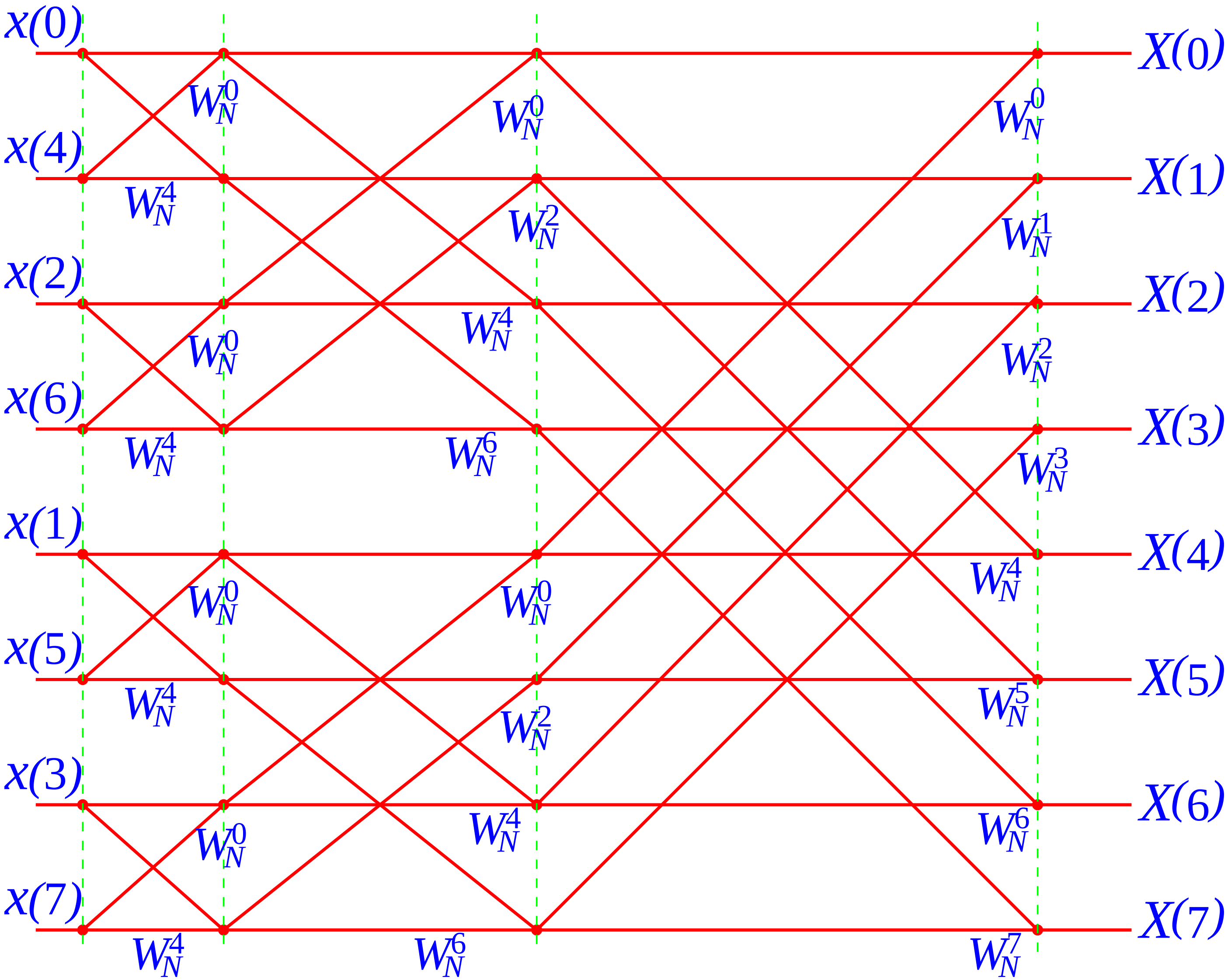
where $G_{N/2}(k)$ and $H_{N/2}(k)$ are the $N/2$-point DFT's involving $x(n)$ with even and odd n, respectively



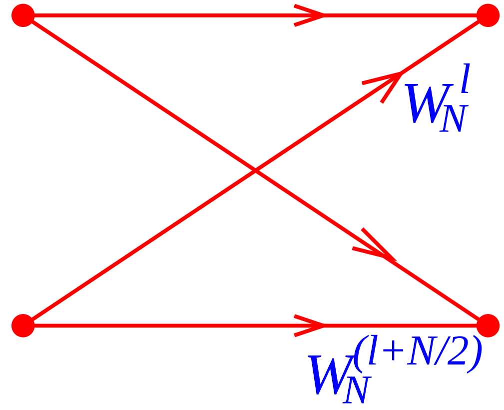
Corollary: any N-point DFT with even N can be computed via two $N/2$-point DFT's. In turn, if $N/2$ is even then each of these $N/2$-point DFT's can be computed via two $N/4$-point DFT's and so on. In the case $N = 2^r$, all $N, N/2, N/4 \ldots$ are even and such a process of "splitting" ends up with 2-point DFT's!



Flow graph of 8-point FFT



Flow graph of basic "butterfly" computation:



Computational complexity: each stage has N complex multiplications and N complex additions and there are $\log_2 N$ stages $\Longrightarrow$ the total complexity (additions and multiplications) is

$$C_{\mathrm{FFT}} = N \log_2 N$$

Compare it with the complexity of the "straightforward" DFT

$$C_{\mathrm{DFT}} = N^2$$

For large N (i.e., $N \gg 1$),

$$\log_2 N \ll N \quad \Longrightarrow \quad C_{\mathrm{FFT}} \ll C_{\mathrm{DFT}}$$

Example: For $N = 2^{10} = 1024$, $C_{\mathrm{DFT}} = 2^{20} \simeq 10^6$, whereas $C_{\mathrm{DFT}} = 10 \cdot 1024 \simeq 10^4$, i.e., the reduction is about 100 orders of magnitude!

6 DIGITAL FILTERS

6.1 What is Filtering

Definition: Digital filtering is just changing the frequency-domain characteristics of a given discrete-time signal

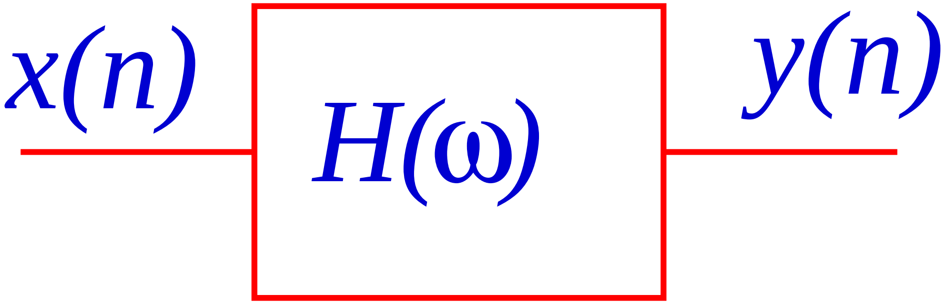
Filtering operations may include:

- noise suppression
- enhancement of selected frequency ranges or edges in images
- bandwidth limiting (to prevent aliasing of digital signals or to reduce interference of neighboring channels in wireless communications)
- removal or attenuation of specific frequencies
- special operations like integration, differentiation, etc.

Recall causal LTI-systems:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \{x(n)\} * \{h(n)\}$$

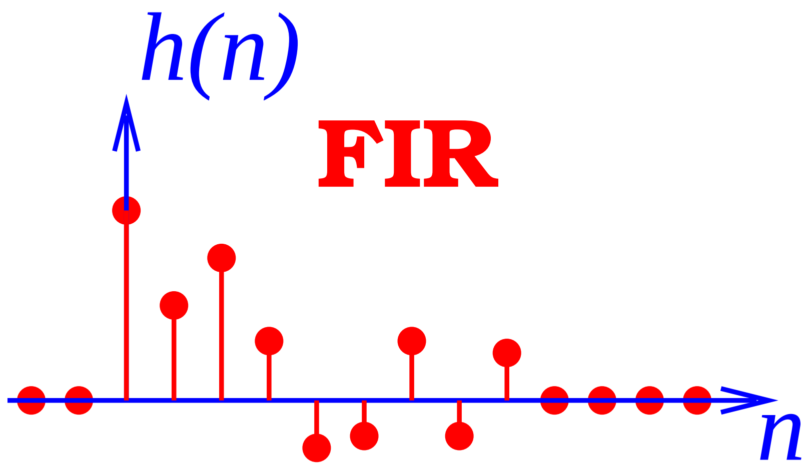
$$Y(\omega) = H(\omega)X(\omega)$$

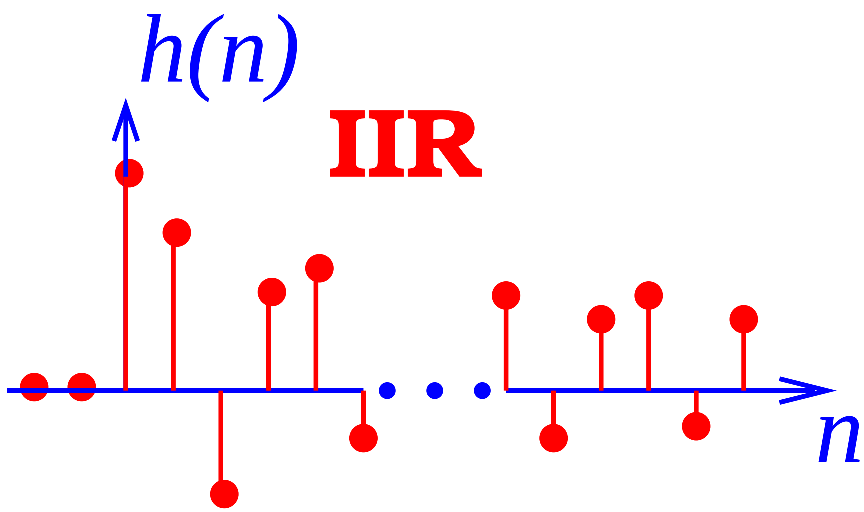


The impulse response values $\{h(0), h(1), h(2), \ldots\}$ can be interpreted as filter coefficients

6.2 Finite and Infinite Impulse Responses

Definition: If $\{h(n)\}$ is an infinite duration sequence, the corresponding filter is called an infinite impulse response (IIR) filter. In turn, if $\{h(n)\}$ is a finite duration sequence, the corresponding filter is called a Finite Impulse Response (FIR) filter.





A very general form of digital filter can be obtained from the familiar equation (recall LTI-systems)

$$\sum_{k=0}^{N} a(k)y(n-k) = \sum_{k=0}^{M} b(k)x(n-k) \qquad \text{ARMA}$$

where $\{x(n)\}$ is the filter input signal and $\{y(n)\}$ is the filter output signal. As obtained before, the transfer function corresponding to this equation is the following rational function

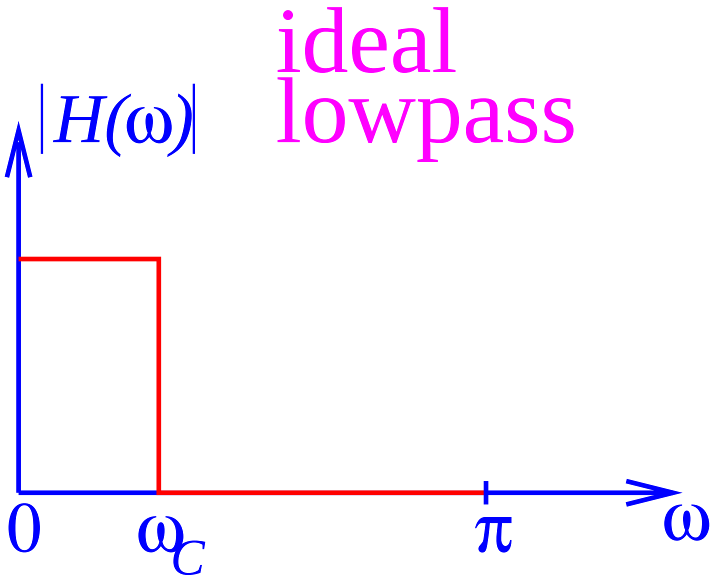
$$H(z) = \frac{\sum_{k=0}^{M} b(k)z^{-k}}{\sum_{k=0}^{N} a(k)z^{-k}} = \frac{B(z)}{A(z)}$$

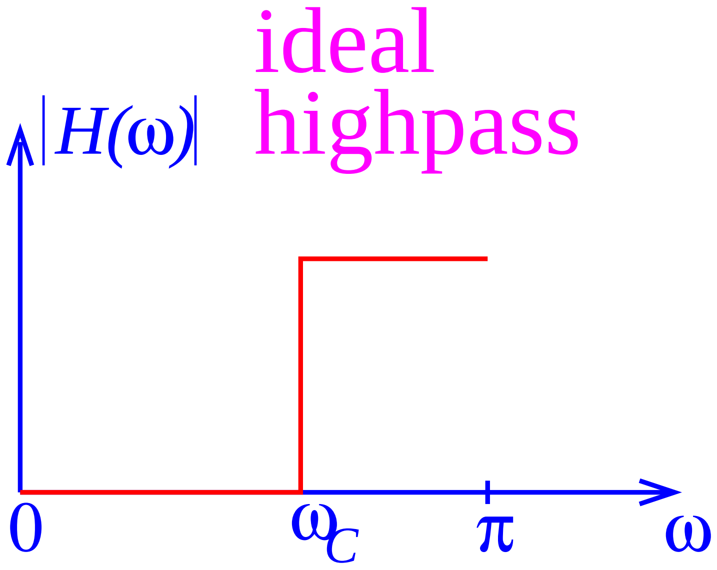
- If $N = 0$, the filter becomes a FIR (nonrecursive) one
- If $N > 0$, the filter becomes a IIR (recursive) one

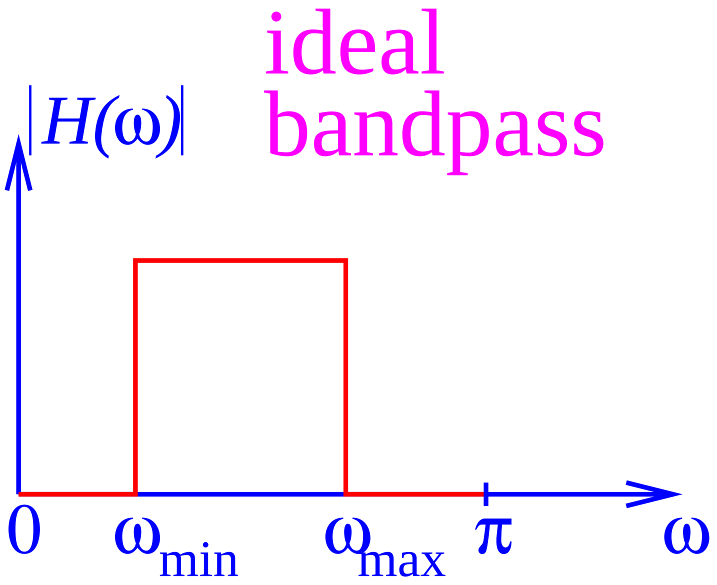
6.3 Filter Specifications

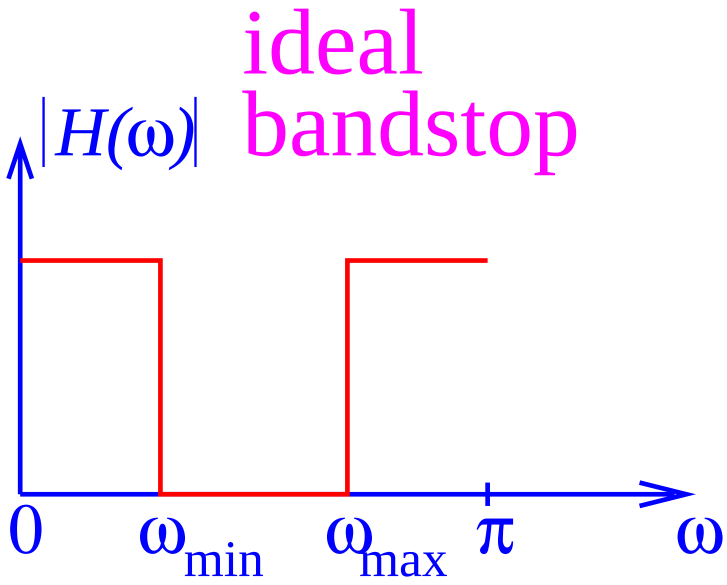
Basic filter types:

- lowpass filters (to pass low frequencies from zero to a certain cut-off frequency ω_C and to block higher frequencies)
- highpass filters (to pass high frequencies from a certain cut-off frequency ω_C to π and to block lower frequencies)
- bandpass filters (to pass a certain frequency range $[\omega_{\mathrm{min}}, \omega_{\mathrm{max}}]$, which does not include zero, and to block other frequencies)
- bandstop filters (to block a certain frequency range $[\omega_{\mathrm{min}}, \omega_{\mathrm{max}}]$, which does not include zero, and to pass other frequencies)


ideal
lowpass
|H(ω)|
0
ω_C
π
ω


ideal
highpass
|H(ω)|
0
ω_C
π
ω

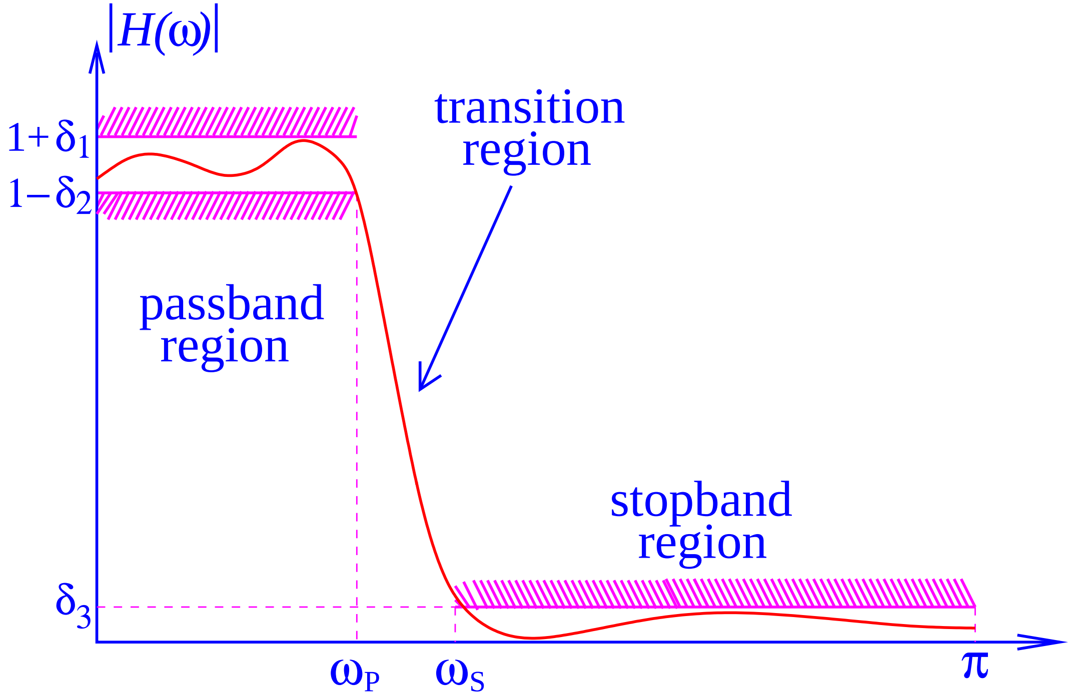

ideal
bandpass
|H(ω)|
0
ω_min
ω_max
π
ω


ideal
bandstop
|H(ω)|
0
ω_min
ω_max
π
ω

Frequency responses of practical filters are not shaped in straight lines, i.e., they vary continuously as a function of frequency: they are neither exactly 1 in the passbands, nor exactly 0 in the stopbands

Lowpass filter specifications:

$$1-\delta_2 \le |H(\omega)| \le 1+\delta_1\,, \omega \in [0,\omega_P]\,; \quad 0 \le |H(\omega)| \le \delta_3\,, \omega \in [\omega_S,\pi]$$



Definition: The quantity $\max\{\delta_1, \delta_2\}$ is called passband ripple, and the quantity δ_3 is called stopband attenuation

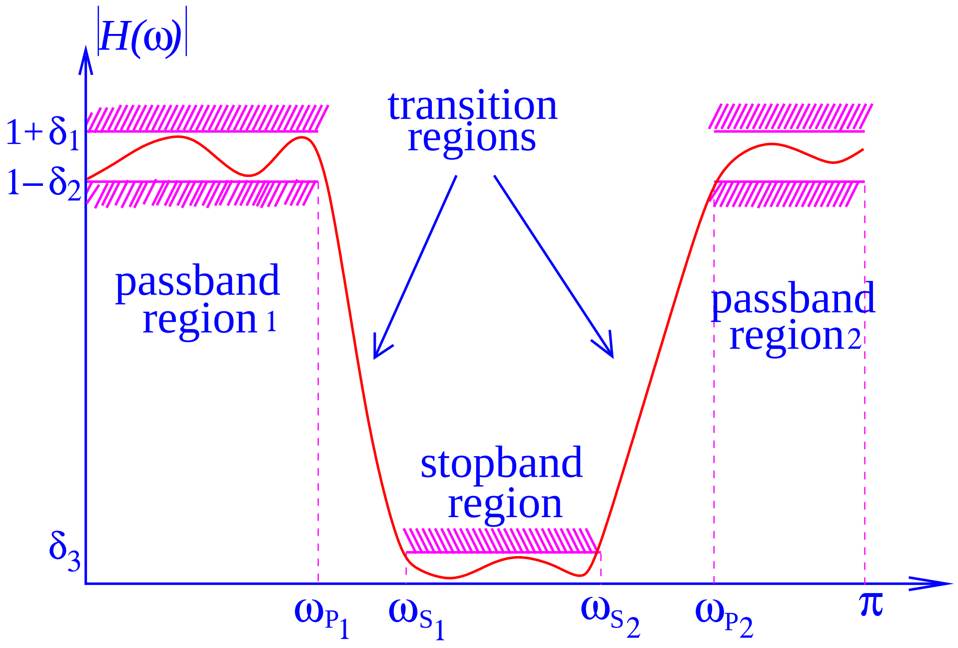
These filter parameters are usually specified in dB:

$$\begin{aligned} A_P &= \max\{20\log_{10}(1+\delta_1), -20\log_{10}(1-\delta_2)\} && \leftarrow \text{PB ripple in dB} \\ A_S &= -20\log_{10}\delta_3 && \leftarrow \text{SB attenuation in dB} \end{aligned}$$

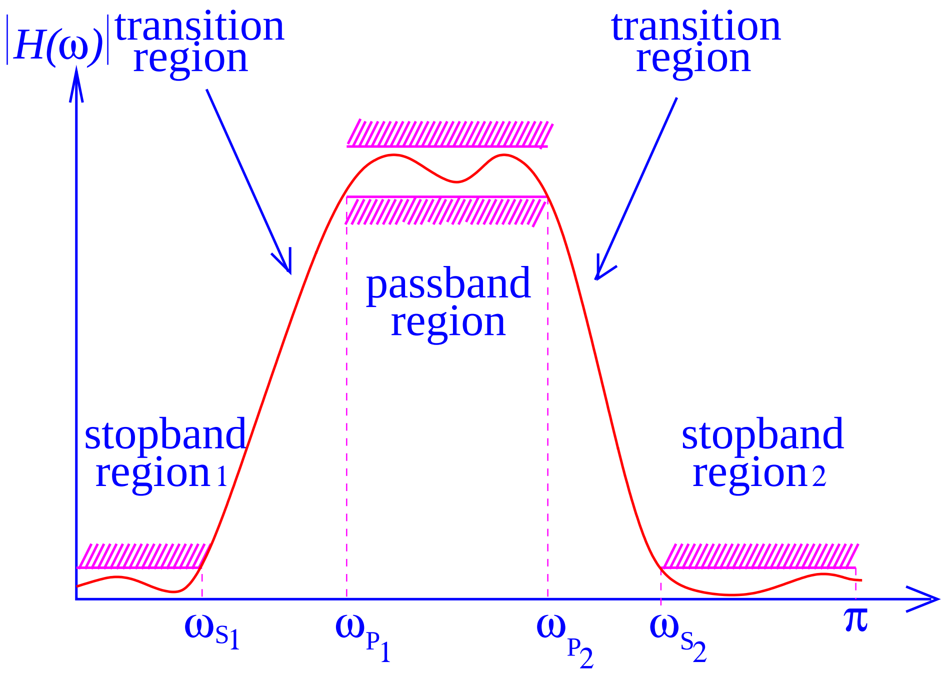
Example: let $\delta_1 = \delta_2 = \delta_3 = 0.1 \quad \Longrightarrow$

$$\begin{aligned} A_P &= \max\{0.828, 0.915\}\ \text{dB} = 0.915\ \text{dB} \\ A_S &= 20\ \text{dB} \end{aligned}$$

BANDSTOP FILTER



BANDPASS FILTER



6.4 The Phase Response and Distortionless Transmission

In most filter applications, the magnitude response $|H(\omega)|$ is of primary concern. However, the phase response may be also important.

$$H(\omega) = |H(\omega)|e^{j\Psi(\omega)}$$

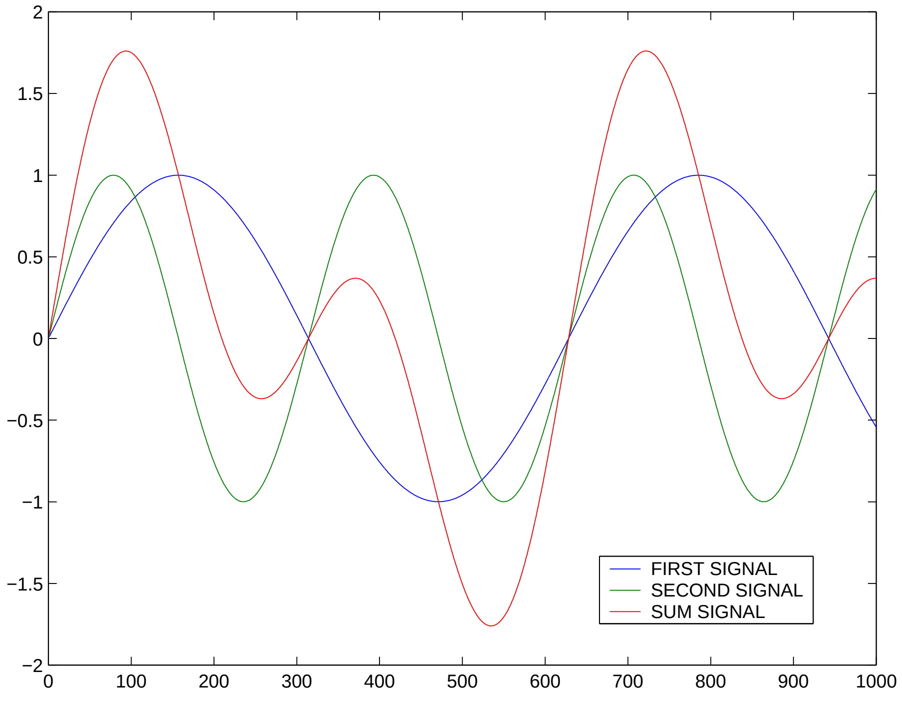
where

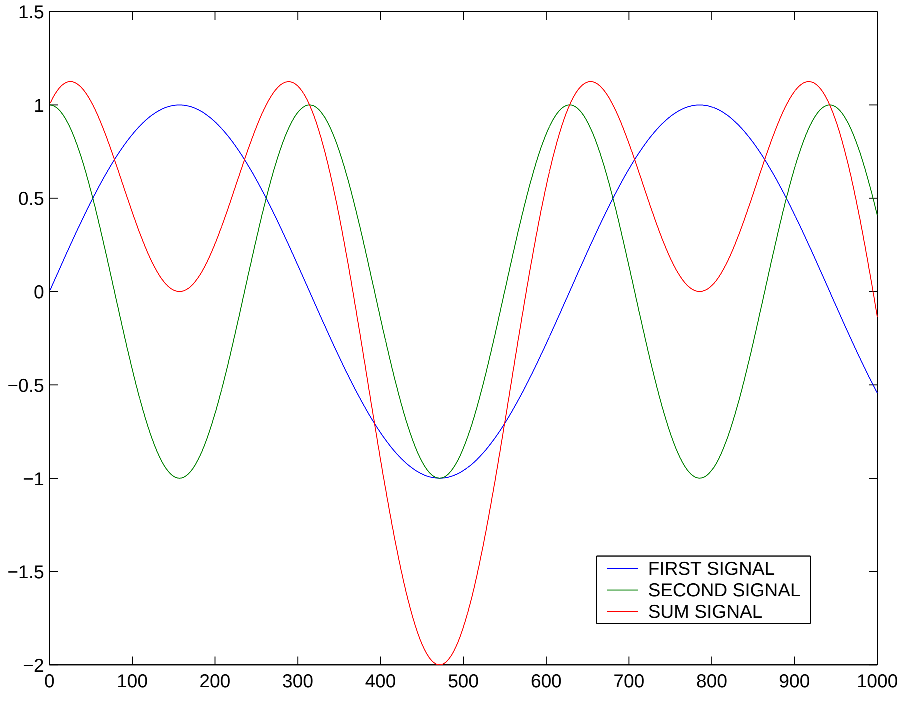
$$\Psi = \text{angle}\{H(\omega)\} \quad \leftarrow \text{phase response}$$

Definition: If a signal is transmitted through a system (filter) then this system is said to provide a distortionless transmission if the signal form remains unaffected, i.e., if the output signal is a delayed and scaled replica of the input signal.

Two conditions of distortionless transmission:

- the system must amplify (or attenuate) each frequency component uniformly, i.e., the magnitude response must be uniform within the signal frequency band.
- the system must delay each frequency component by the same value.





Definition: The group delay of a filter is the relative delay imposed on a particular frequency component of an input signal.

We can represent an arbitrary phase response in the form

$$\Psi(\omega) = -\omega\tau(\omega)$$

where $\tau(\omega)$ is the group delay measured in samples.

Result: To satisfy the distortionless response phase condition, the group delay must be frequency-independent, i.e., uniform for each frequency $\Longrightarrow$

$$\tau(\omega) = \tau = \text{const}$$

A filter having this property is called a linear-phase filter because phase varies linearly with the frequency ω

Hence in general, a linear-phase frequency response is given by

$$H(\omega) = |H(\omega)|\, e^{-j\omega\tau}$$

Example: Ideal lowpass with linear phase

$$H(\omega) = \begin{cases} e^{-j\omega\tau}\,, & |\omega| < \omega_C \\ 0\,, & \omega_C < |\omega| < \pi \end{cases}$$

The corresponding impulse response

$$h(n) = \frac{\sin\{\omega_C(n-\tau)\}}{\pi(n-\tau)}$$

Taking an integer delay $\tau = m$, we have

$$h(2m-n) = \frac{\sin\{\omega_C(2m-n-m)\}}{\pi(2m-n-m)} = \frac{\sin\{\omega_C(m-n)\}}{\pi(m-n)} = h(n)$$

Hence, the response is symmetric about $n = m$ $\Longrightarrow$ in that case we can define a zero-phase system

$$\tilde{H}(\omega) = H(\omega)\, e^{j\omega m} = |H(\omega)| \;, \quad \tilde{h}(n) = \frac{\sin(\omega_C n)}{\pi n}$$

Definition: A system is referred to as a generalized linear-phase system if its frequency response can be expressed in the form

$$H(\omega) = A(\omega)\, e^{j(\alpha - \omega\tau)} \qquad (*)$$

with a real function $A(\omega)$ and $\tau = \text{const}$, $\alpha = \text{const}$.

Example: consider a discrete-time implementation of the ideal continuous-time differentiator

$$y_c(t) = \frac{d}{dt} x_c(t) \quad \Longrightarrow \quad H(\omega) = j\omega$$

Restrict the input to be bandlimited

$$\tilde{H}_c(\omega_c) = \begin{cases} j\omega_c\,, & |\omega_c| < \pi/\Delta t \\ 0\,, & |\omega_c| \geq \pi/\Delta t \end{cases}$$

The corresponding discrete-time system has frequency response

$$H(\omega) = \frac{j\omega}{\Delta t}\,, \quad |\omega| < \pi$$

Hence, $H(\omega)$ has the form $(*)$ with $\tau = 0$, $\alpha = \pi/2$, and $A(\omega) = \omega/\Delta t \implies$ This system is a generalized linear-phase filter!

6.5 FIR Filters

The transfer function:

$$
\begin{aligned}
H(\omega) &= h(0) + h(1)e^{-j\omega} + \cdots + h(N-1)e^{-j\omega(N-1)} \\
&= \sum_{n=0}^{N-1} h(n)e^{-j\omega n} \qquad \textbf{FIR}
\end{aligned}
$$

Advantages of FIR Filters:

- FIR filters are stable
- they can be designed to have linear phase or generalized linear phase
- there is great possibility in shaping their magnitude response
- they are convenient to implement

Impulse response truncation method:

Step 1: Chose the desired amplitude response according to filter class (lowpass, highpass, etc.)

Step 2: Chose the filters phase characteristics: integer or fractional group delay, initial phase

Step 3: Write the ideal frequency response as $H_{\text{id}}(\omega) = A_{\text{id}}(\omega)\exp\{j(\alpha_{\text{id}} - \omega\tau_{\text{id}})\}$ and compute the ideal impulse response $h_{\text{id}}(n)$ using the inverse DTFT

Step 4: Truncate the impulse response by taking

$$h(n) = \begin{cases} h_{\text{id}}(n)\,, & 0 \leq n \leq N-1 \\ 0\,, & n > N-1 \end{cases}$$

Example: Let the lowpass ideal frequency response be

$$H(\omega) = \begin{cases} e^{-j0.5\omega N}, & 0 \le \omega \le \pi/4 \\ 0, & \text{otherwise} \end{cases}$$

or, in terms of f,

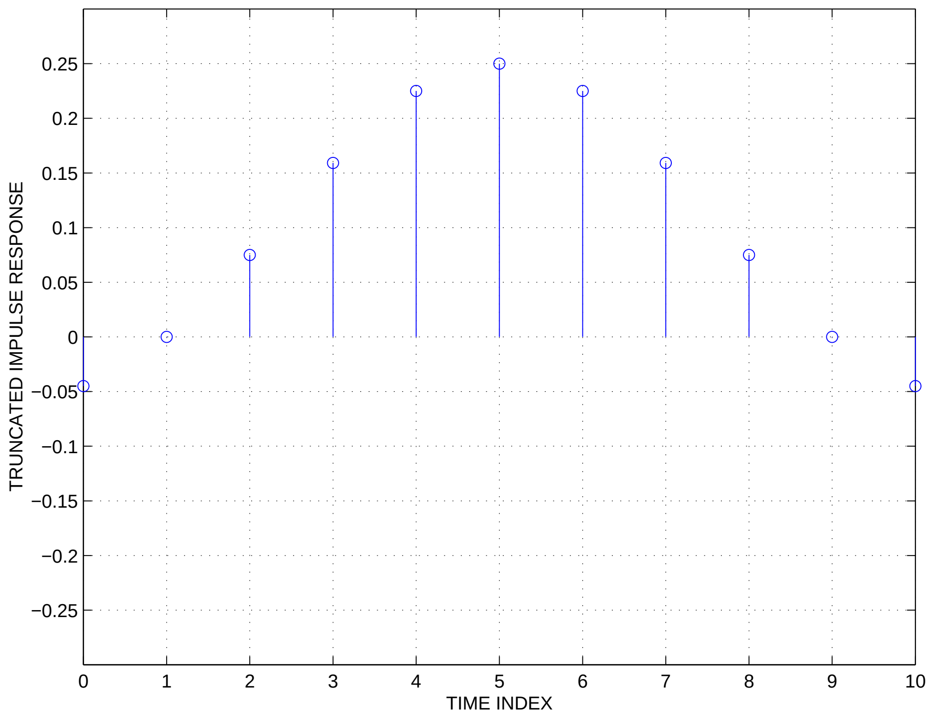
$$H(f) = \begin{cases} e^{-j\pi f N}, & 0 \le f \le 0.125 \\ 0, & \text{otherwise} \end{cases}$$

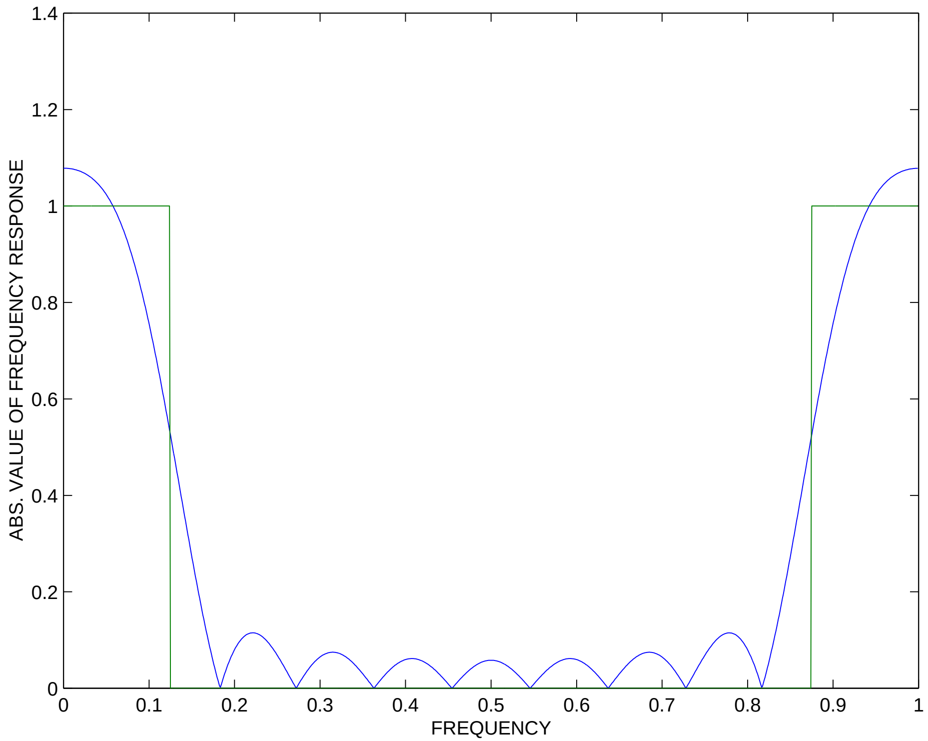
Then, the ideal impulse response is

$$h_{\text{id}}(n) = \frac{\sin(\pi(n - 0.5N)/4)}{\pi(n - 0.5N)}$$

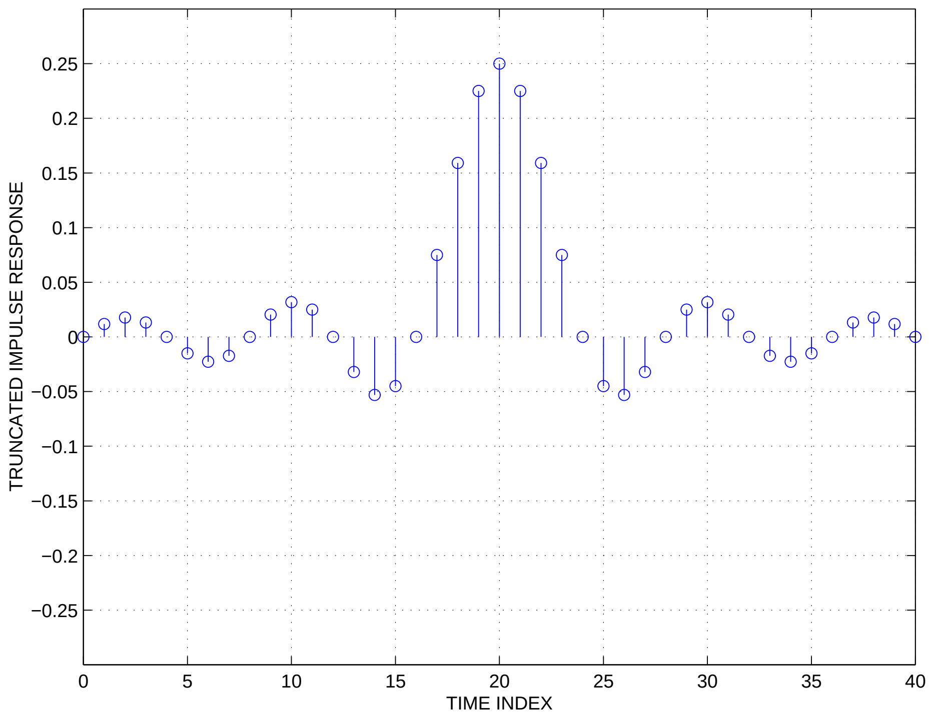
Consider different filter orders $N = 11$, $N = 41$, and $N = 161$

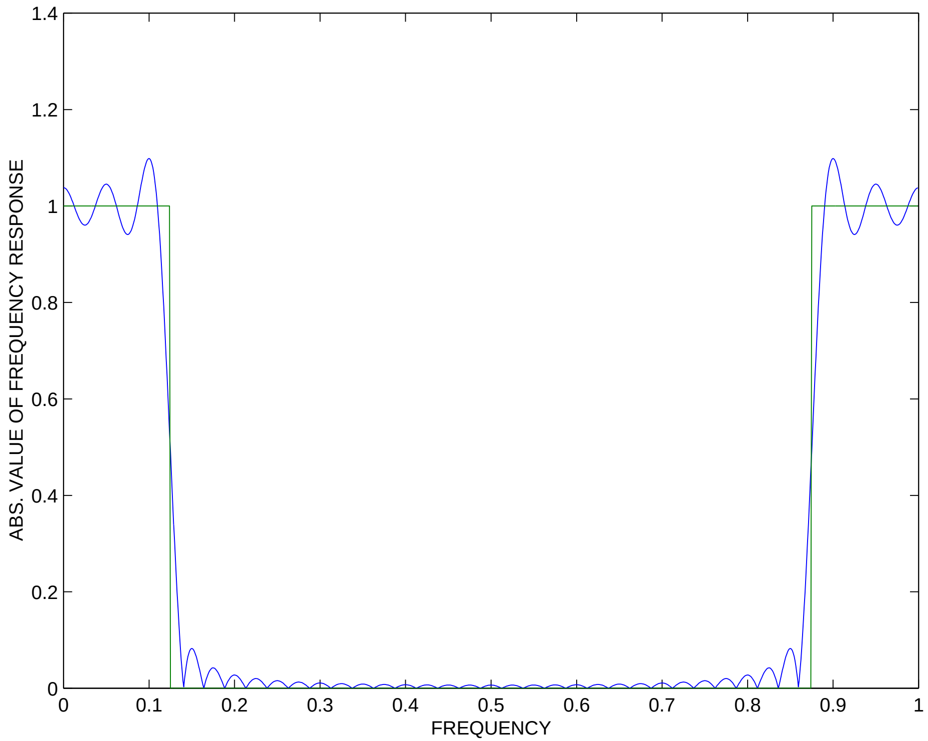
Truncated impulse response ($N = 11$) and corresponding approximation of lowpass frequency response



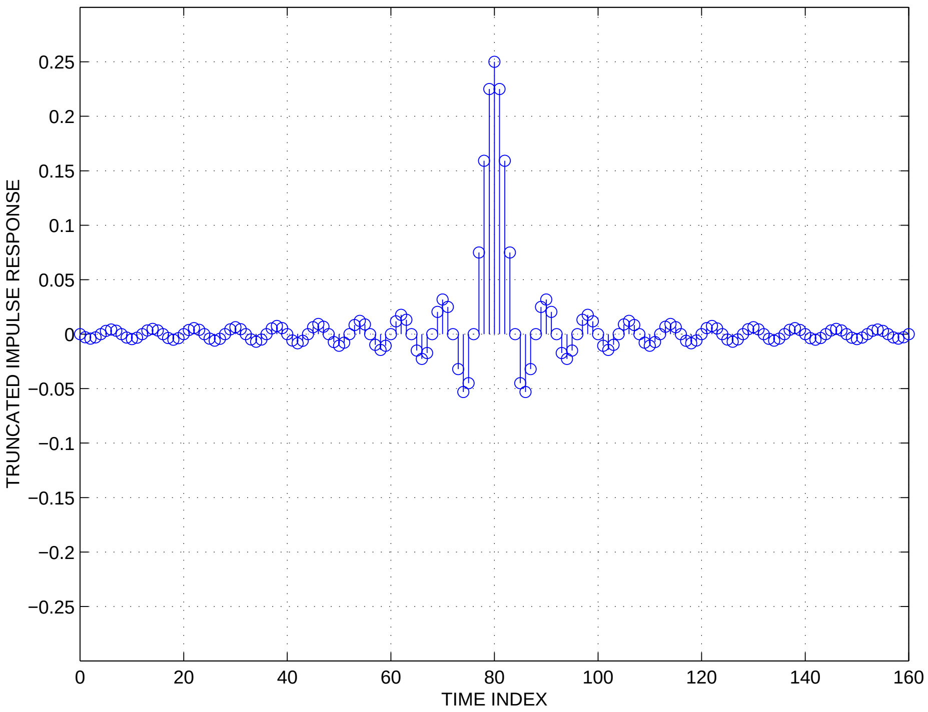


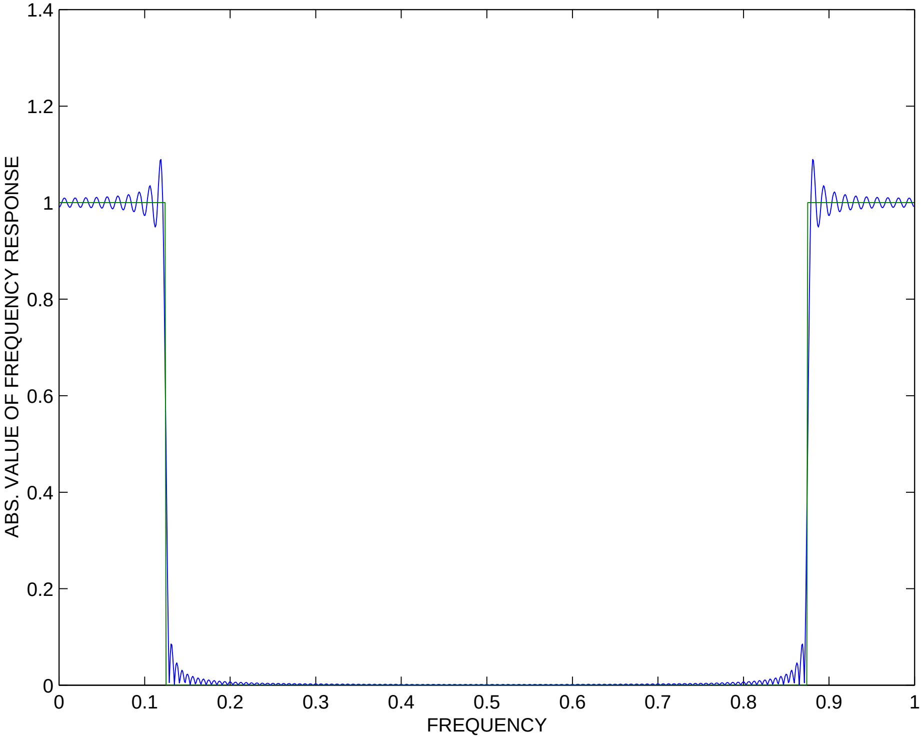
Truncated impulse response $(N = 41)$ and corresponding approximation of lowpass frequency response



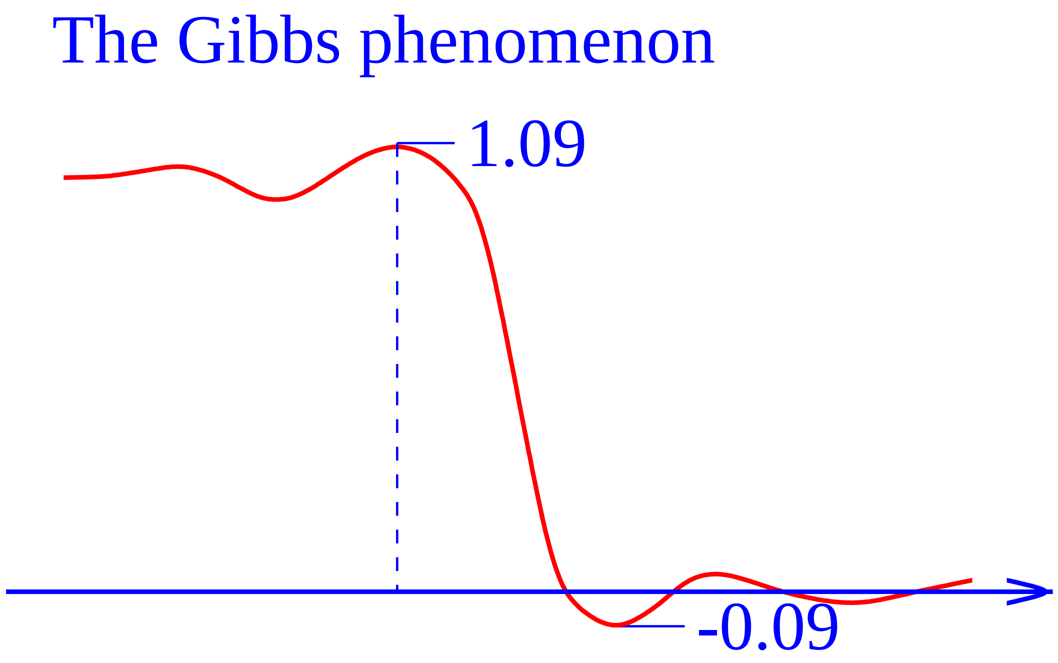


Truncated impulse response ($N = 161$) and corresponding approximation of lowpass frequency response





Gibbs phenomenon: oscillations at the edges of the pass- and stopband cannot be reduced by the increase of N! For any N, these oscillations correspond to about 0.09 passband ripple and stopband attenuation parameters! $\Longrightarrow$ the impulse response truncation method is easy but not a very good way of filter design



FIR filter design using windows:

Step 1: Define the ideal frequency response as in the impulse response truncation method

Step 2: Obtain the impulse response $h_{\mathrm{id}}(n)$ of this ideal filter as in the impulse response truncation method

Step 3: Compute the coefficients of the filter by

$$h(n) = \begin{cases} w(n)h_{\mathrm{id}}(n)\,, & 0 \leq n \leq N-1 \\ 0\,, & n > N-1 \end{cases}$$

where $w(n)$ is some chosen window function

Window properties:

- windows are always real and symmetric, i.e., they must satisfy

$$w(n) = w(N - n - 1), \quad n = 0, \ldots, N - 1$$

for either even or odd N. If the ideal FIR filter has linear phase, symmetric windows do not affect this property.

- windows must be positive to avoid any changes of sequence polarities
- a "good" window must satisfy the tradeoff between the width of the mainlobe and the magnitudes of the sidelobes in the frequency domain
- a "good" window must have a smooth transition at the edges of sequence in order to reduce the truncation effect
- the window length must correspond to the sequence length

Classical window types (have been derived based on intuition and educated guesses):

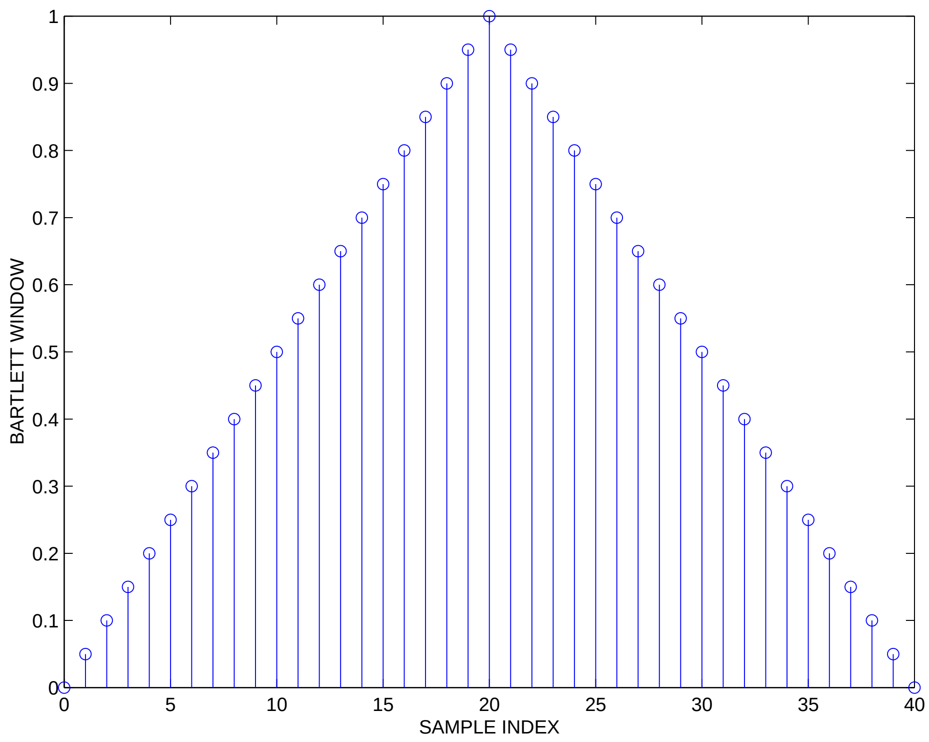
- Rectangular window (corresponds to the natural truncation like in the impulse response truncation method!)

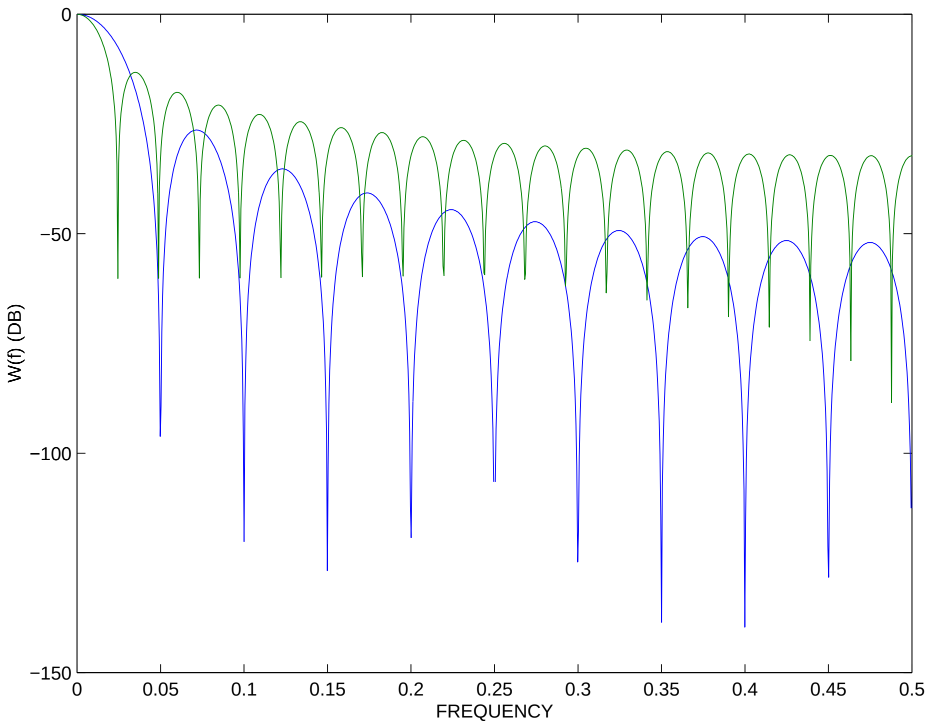
$$w(n) = \begin{cases} 1\,, & 0 \leq n \leq N-1 \\ 0\,, & \text{otherwise} \end{cases}$$

- Triangular (or Bartlett) window

$$w(n) = \begin{cases} 2n/(N-1)\,, & 0 \leq n \leq (N-1)/2 \\ 2 - 2n/(N-1)\,, & (N-1)/2 < n \leq (N-1) \\ 0\,, & \text{otherwise} \end{cases}$$

Bartlett window compared to rectangular window in frequency domain $(N = 41)$





$$W(f)[\mathrm{dB}] = 10\log_{10}\left\{|W(f)|^2\right\} = 20\log_{10}\{|W(f)|\}$$

- Hanning (or Hann) window

$$w(n) = \begin{cases} 0.5\,(1 - \cos[2\pi n/(N-1)])\,, & 0 \le n \le N-1 \\ 0\,, & \text{otherwise} \end{cases}$$

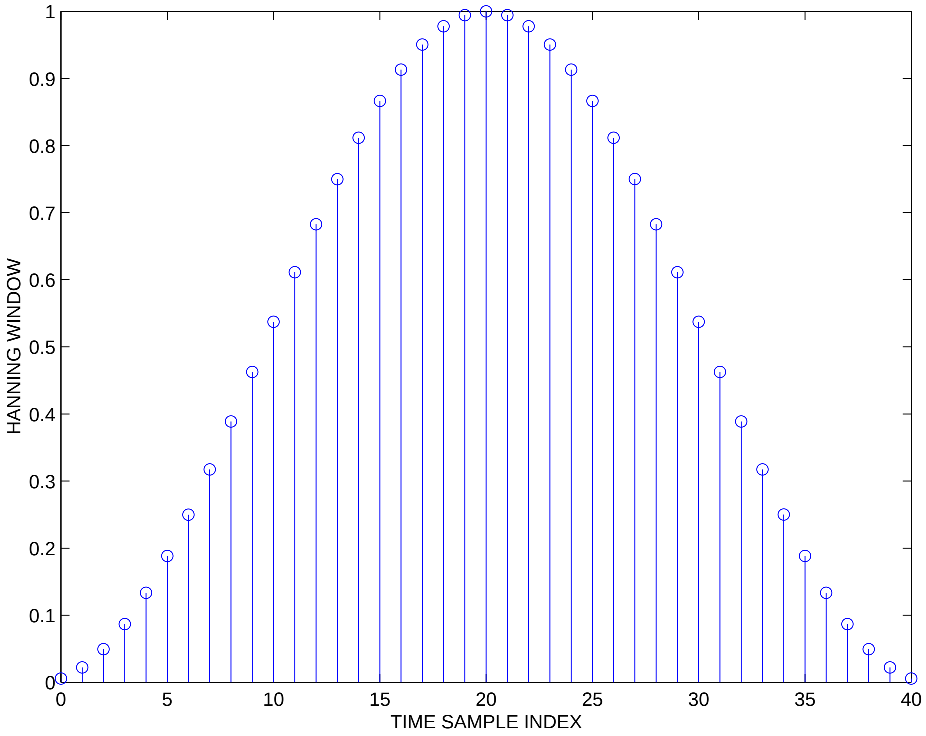
- Hamming window

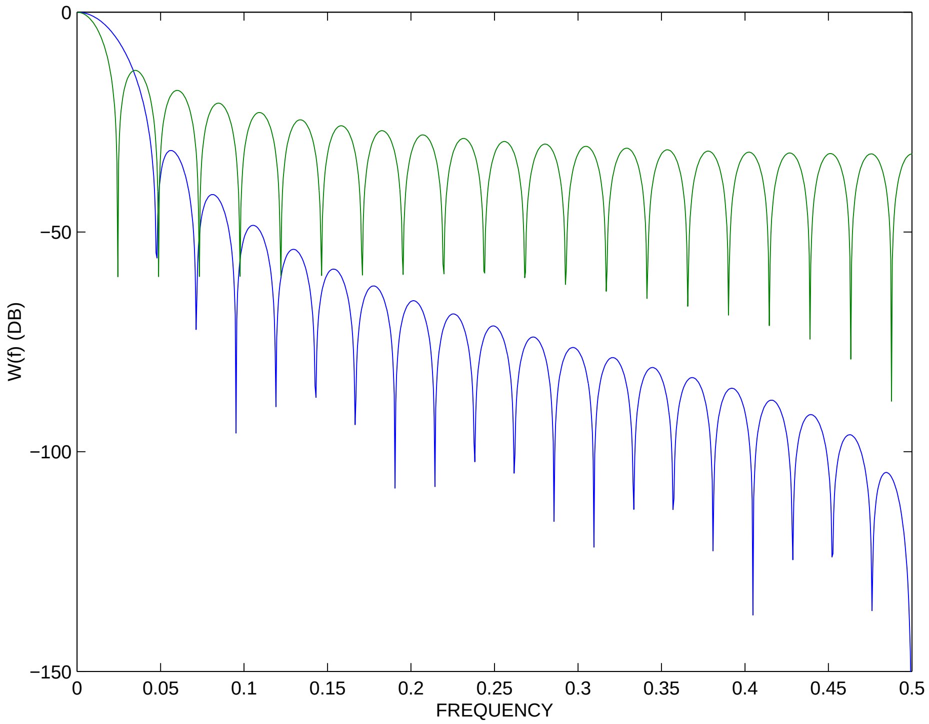
$$w(n) = \begin{cases} 0.54 - 0.46\cos[2\pi n/(N-1)]\,, & 0 \le n \le N-1 \\ 0\,, & \text{otherwise} \end{cases}$$

- Blackman window

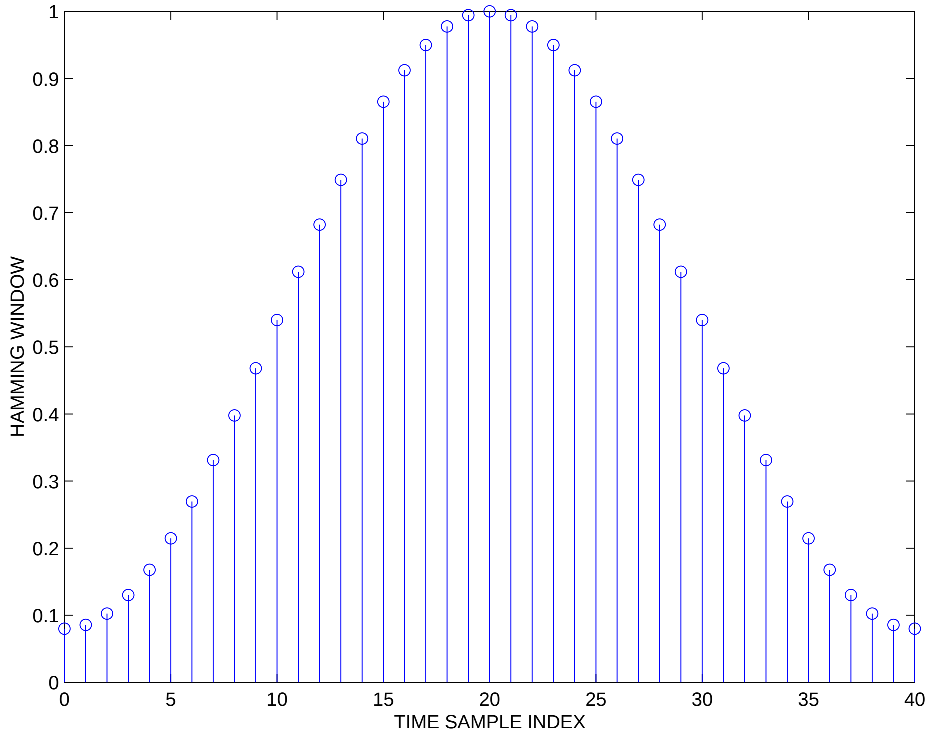
$$w(n) = \begin{cases} 0.42 - 0.5\cos\left[\frac{2\pi n}{N-1}\right] + 0.08\left[\frac{4\pi n}{N-1}\right]\,, & 0 \le n \le N-1 \\ 0\,, & \text{otherwise} \end{cases}$$

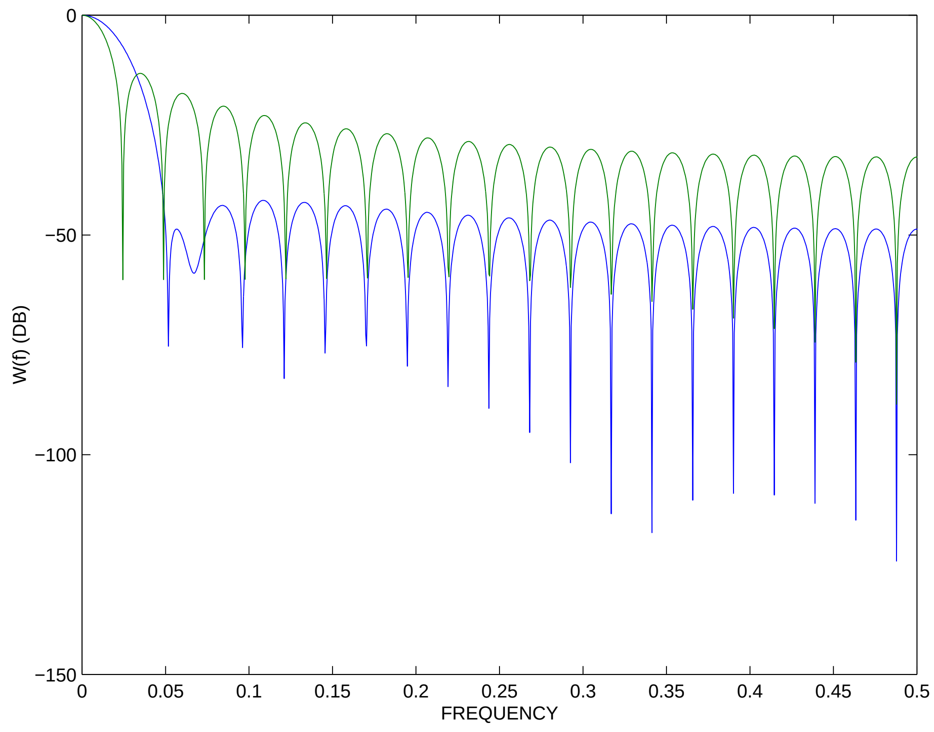
Hanning window compared to rectangular window in frequency domain $(N = 41)$



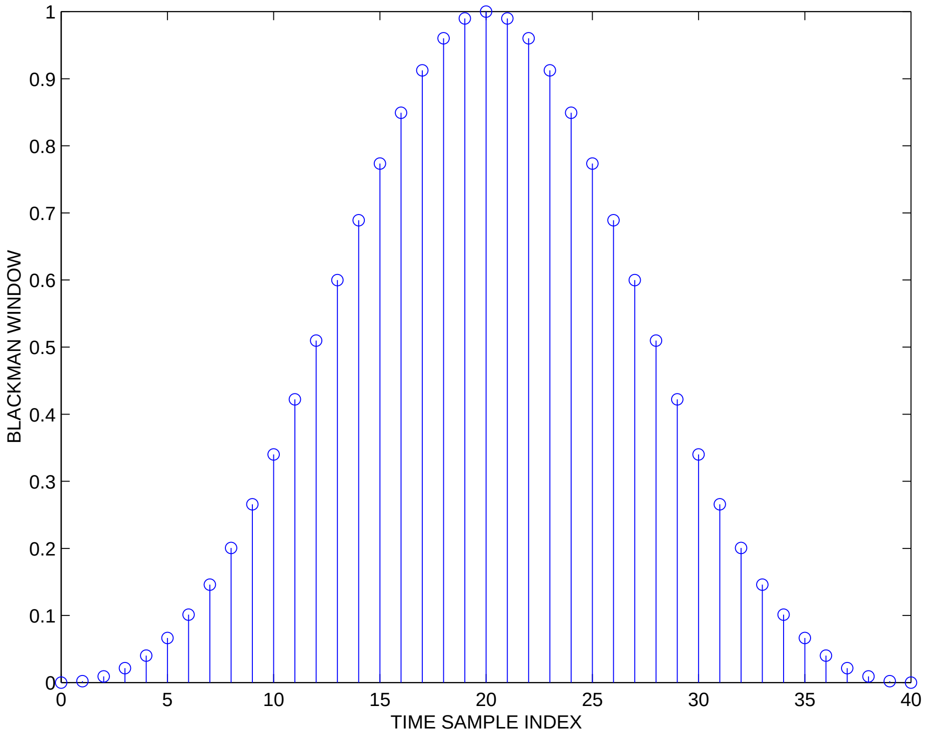


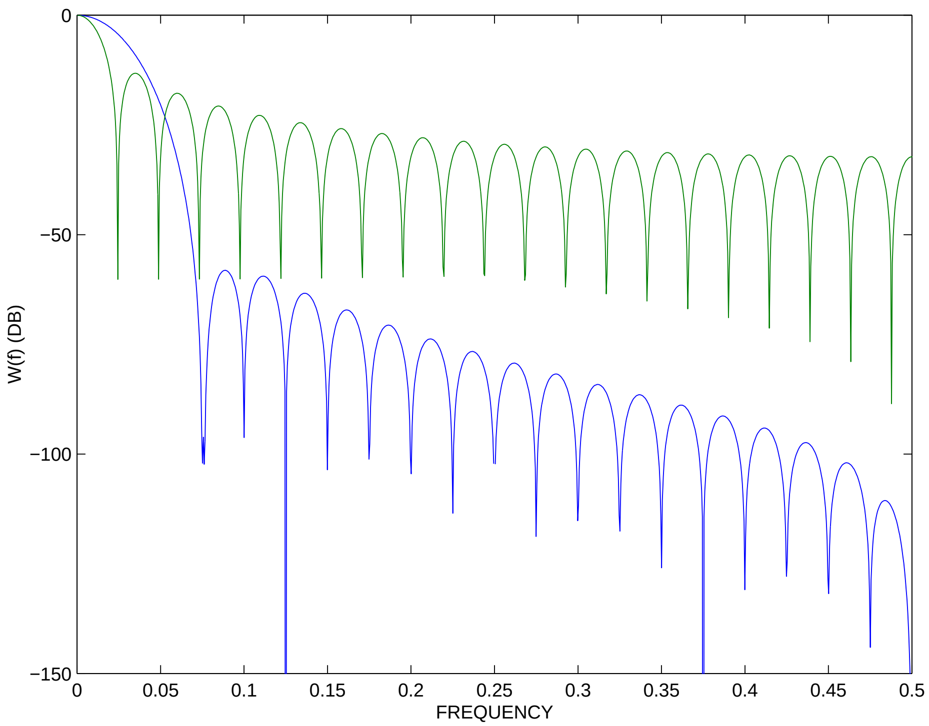
Hamming window compared to rectangular window in frequency domain ($N = 41$)





Blackman window compared to rectangular window in frequency domain $(N = 41)$



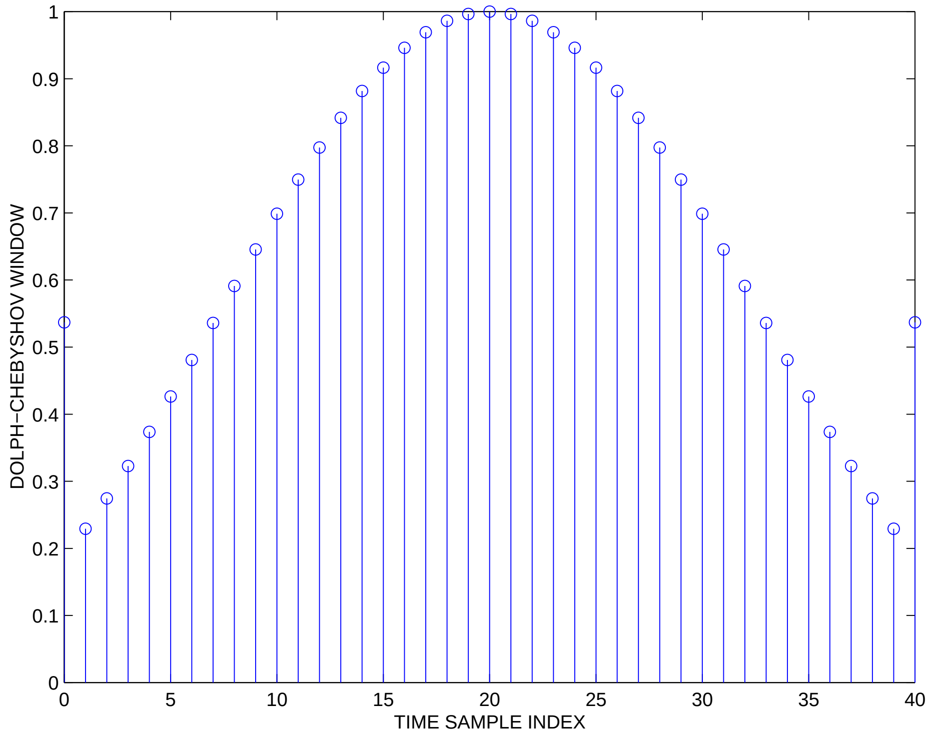


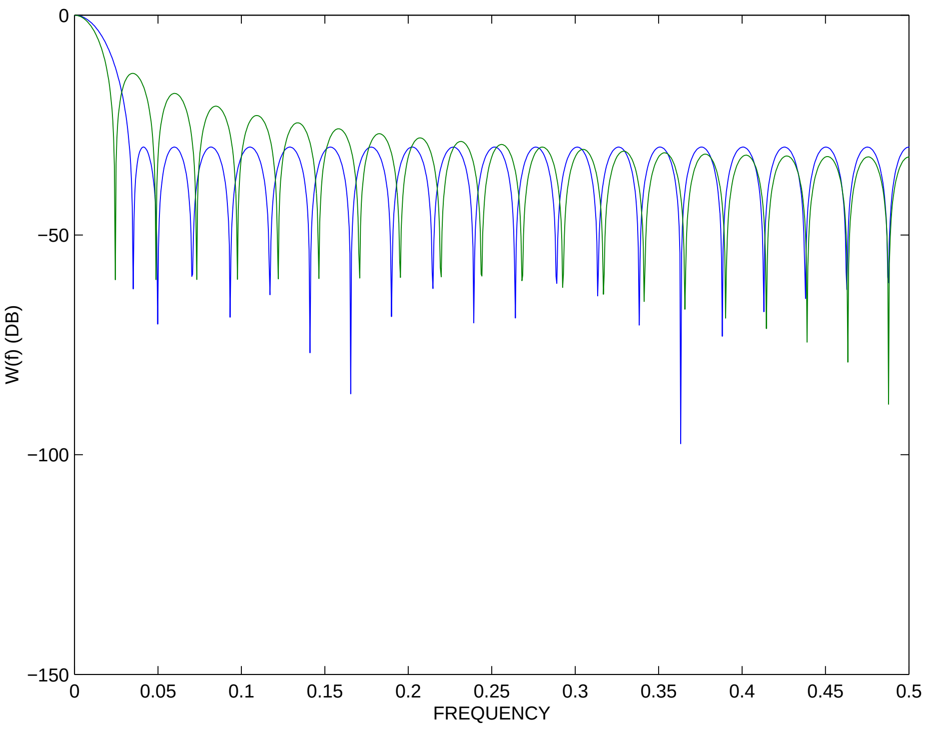
Modern window types (have been derived based on optimality criteria):

- Kaiser window: minimize the width of the mainlobe under the constraint that the window length be fixed and the energy in the sidelobes not exceed a given percentage of the total energy.
- Dolph-Chebyshev window: minimize the width of the mainlobe under the constraint that the window length be fixed and the sidelobe level not exceed a given value.

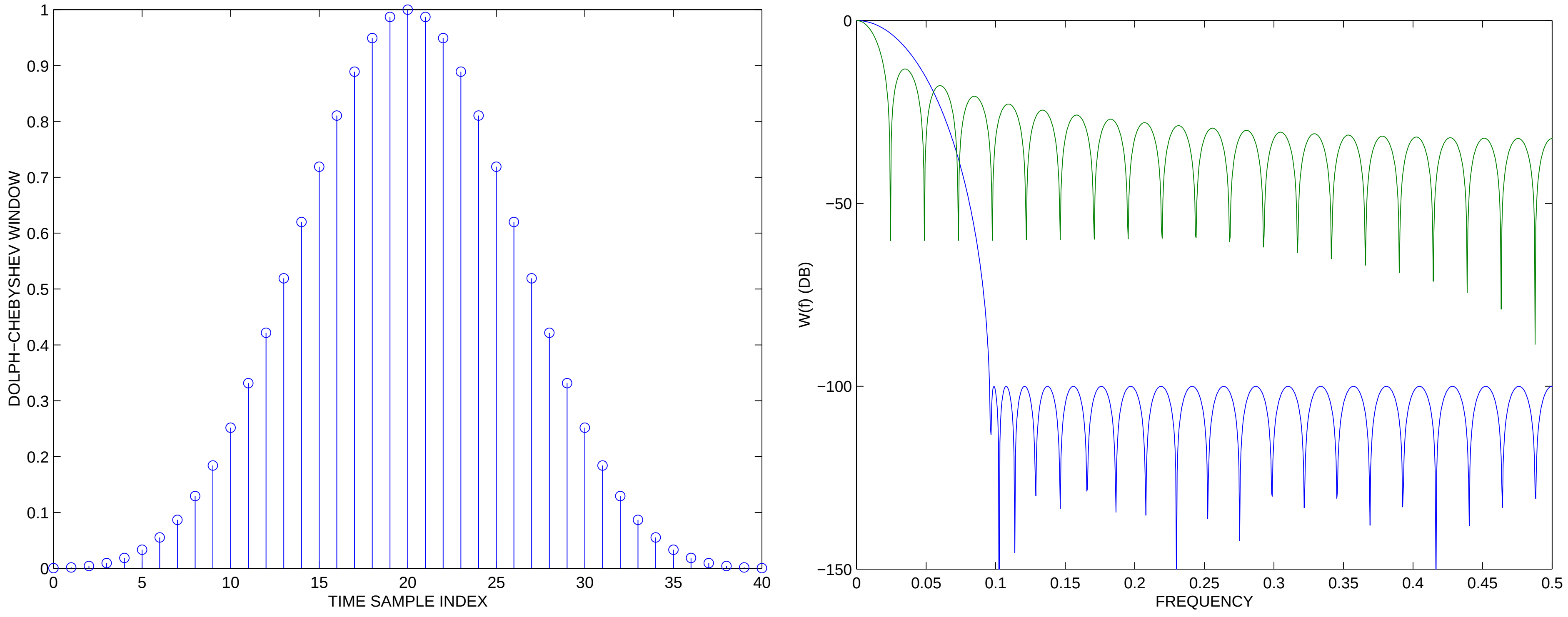
Kaiser and Dolph-Chebyshev windows provide more flexibility than the classical windows because the required tradeoff between the mainlobe and sidelobes can be achieved!

Dolph-Chebyshev window with -30 dB's of ripple compared to the rectangular window in frequency domain $(N = 41)$





Dolph-Chebyshev window with -100 dB's of ripple compared to the rectangular window in frequency domain ($N = 41$)

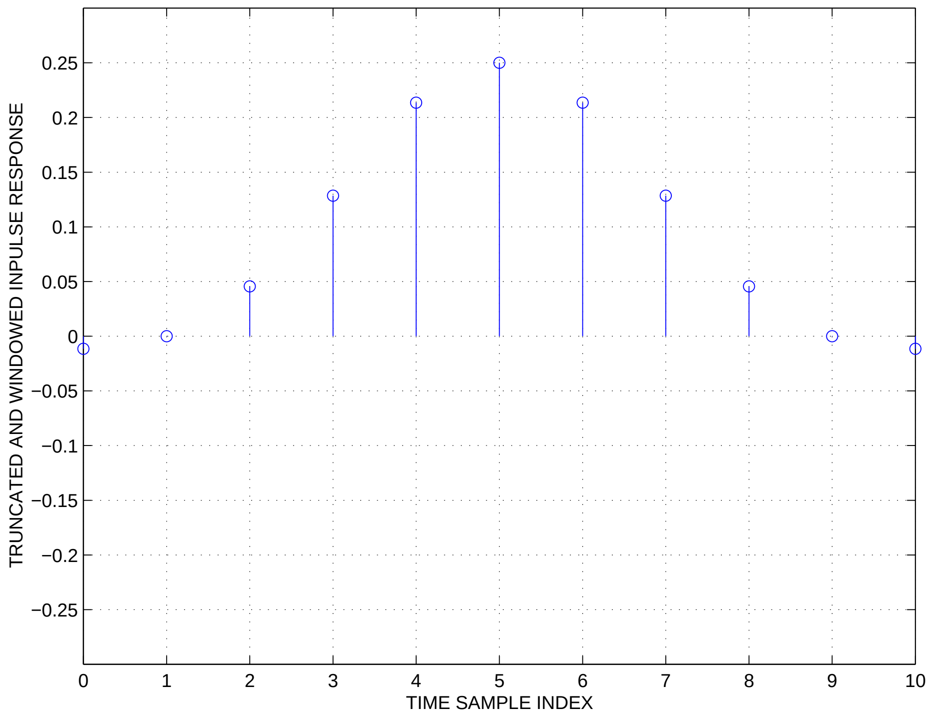


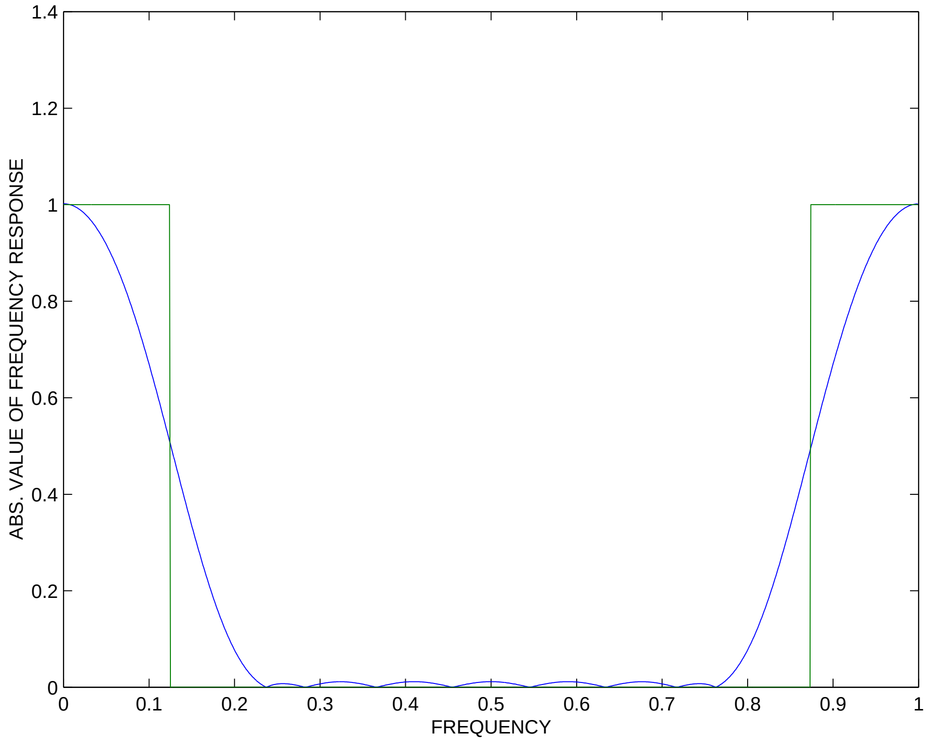
Now, let us take Dolph-Chebyshev window and design the lowpass filter using the window method with $N = 11$, $N = 41$, and $N = 161$.
Remember that

$$h_{\mathrm{id}}(n) = \frac{\sin(\pi(n - 0.5N)/4)}{\pi(n - 0.5N)}$$

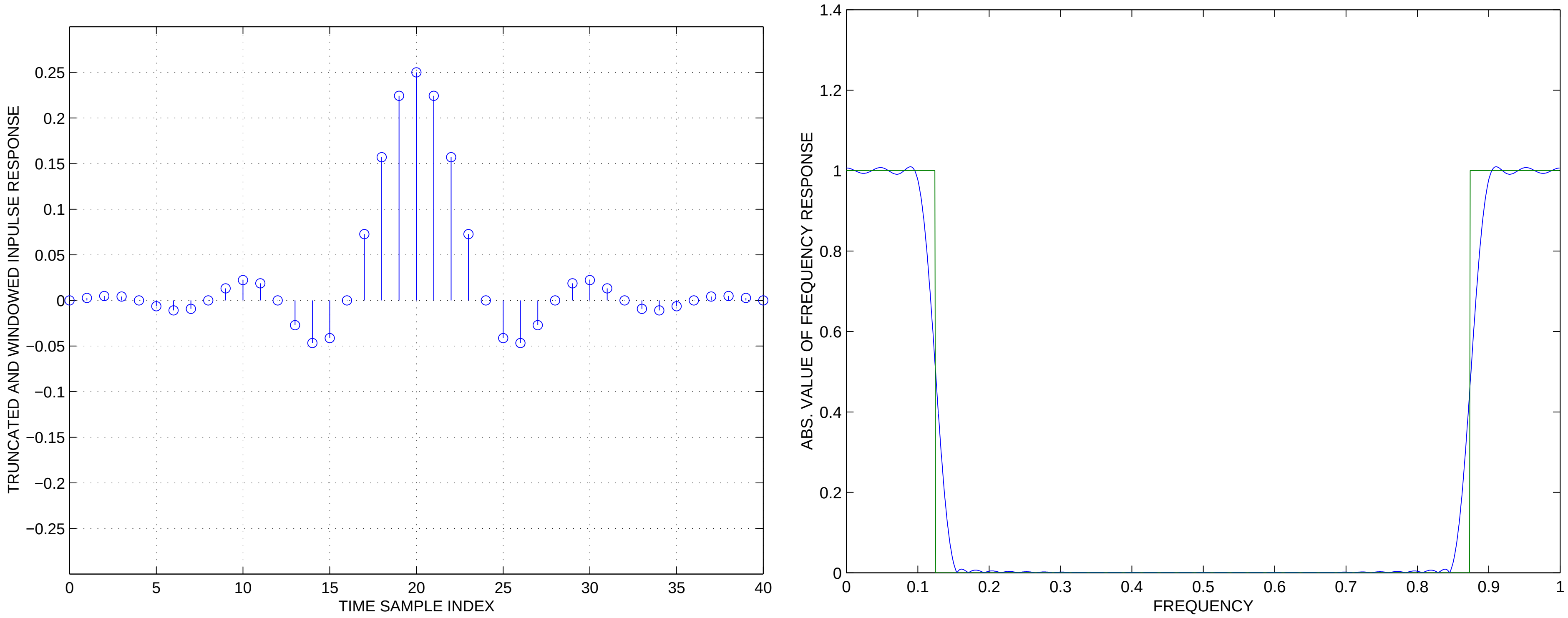
We take the window length equal to the length of $h_{\mathrm{id}}(n)$.

Truncated and windowed impulse response ($N = 11$) and corresponding approximation of lowpass frequency response (window ripple -30 dB)

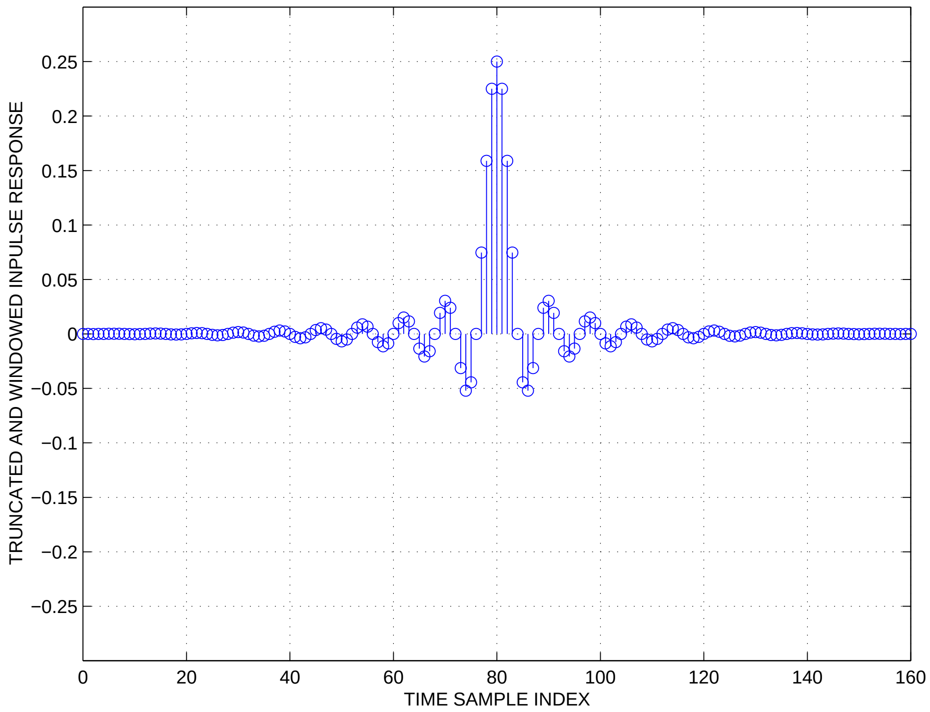


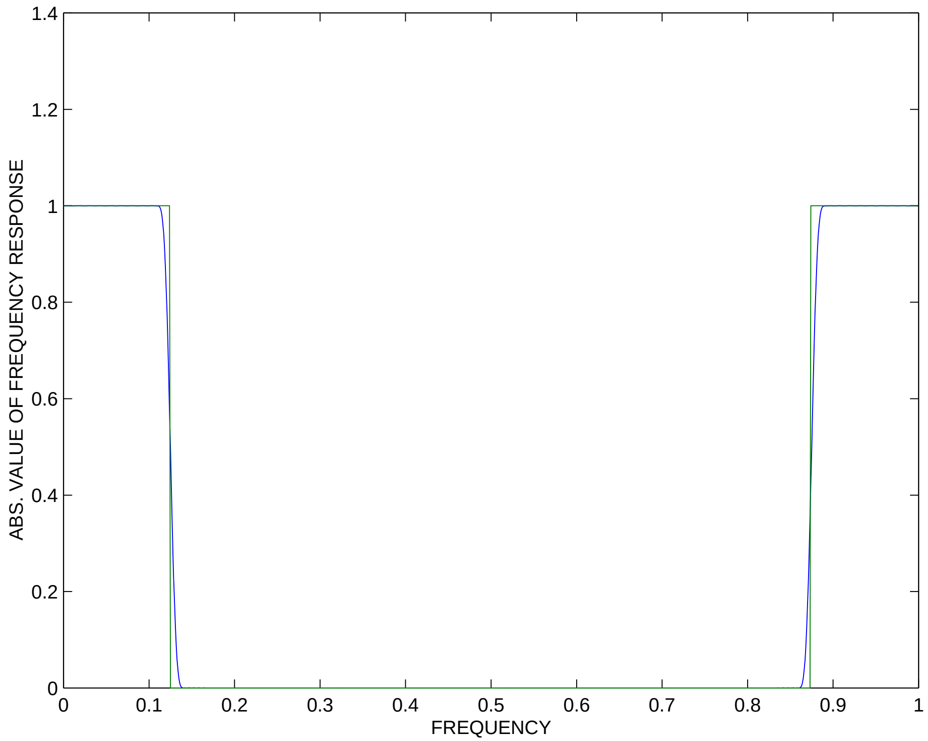


Truncated and windowed impulse response ($N = 41$) and corresponding approximation of lowpass frequency response (window ripple -30 dB)



Truncated and windowed impulse response ($N = 161$) and corresponding approximation of lowpass frequency response (window ripple -60 dB)





There is no Gibbs phenomenon!

Optimization-based methods: the idea is to find the best approximation to the ideal frequency response for a given fixed N, i.e. to find

$$h(n) = \begin{cases} h_{\mathrm{d}}(n)\,, & 0 \leq n \leq N-1 \\ 0\,, & n > N-1 \end{cases}$$

which minimizes the squared error

$$\epsilon^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_{\mathrm{d}}(\omega) - H_{\mathrm{id}}(\omega)|^2 \, d\omega$$

Another powerful criterion is the so called minimax criterion: optimize the impulse response by minimizing the maximal error between the ideal and designed frequency responses.

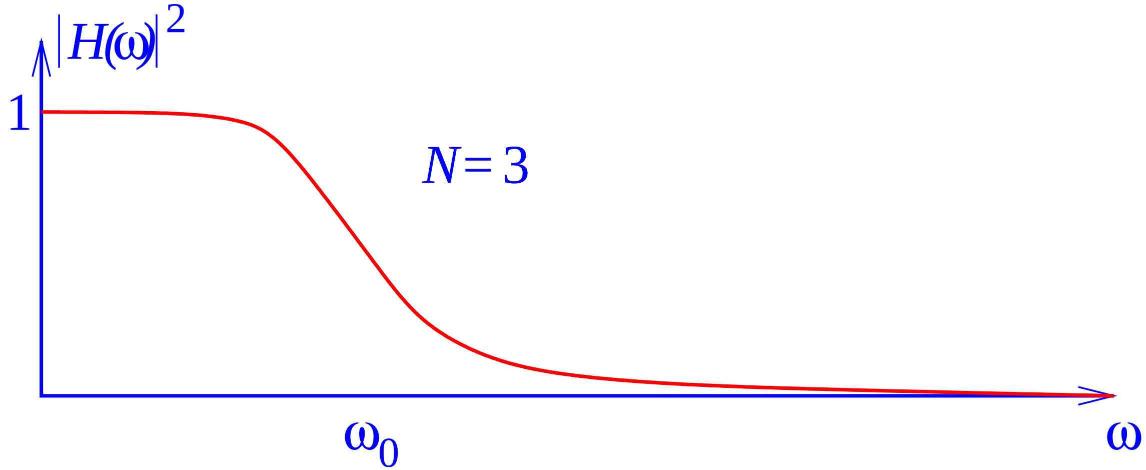
6.6 IIR Filters

We start with analog IIR filters and then find ways to transform a given analog IIR filter to a similar digital filter.

A lowpass Butterworth filter is defined as:

$$|H(\omega)|^2 = \frac{1}{1 + (\omega/\omega_0)^{2N}} \qquad (*)$$

where N is the integer which will soon be seen to be the filter order



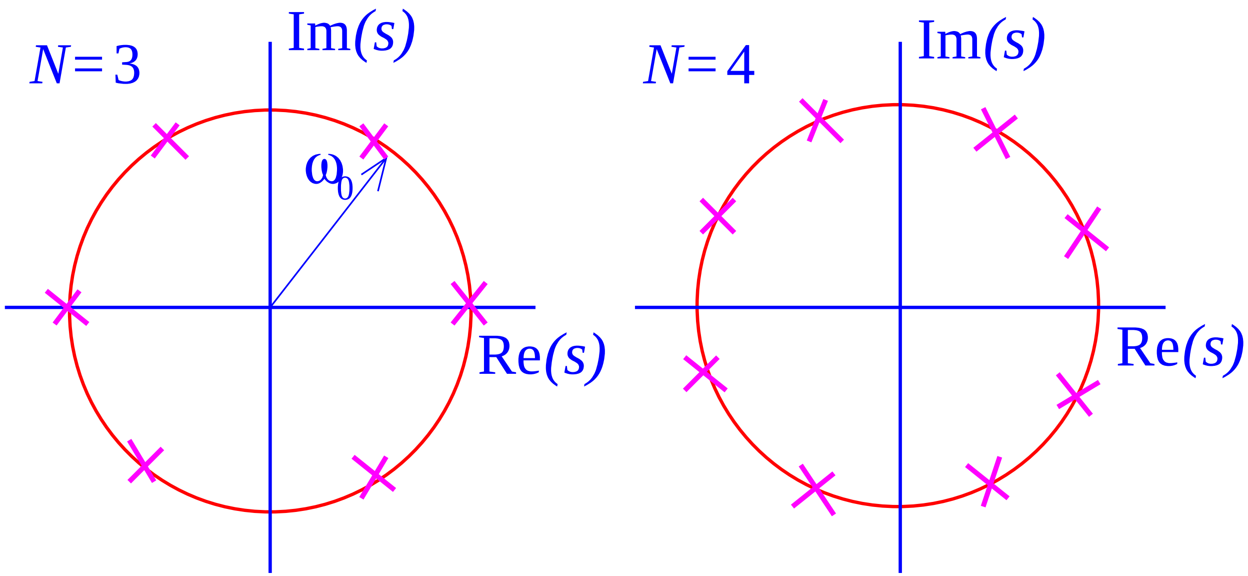
To obtain the transfer function of the Butterworth filter, define $s = j\omega$ and substitute it to $(*)$

$$H(s)H(-s) = \frac{1}{1 + (s/j\omega_0)^{2N}} = \frac{1}{1 + (-1)^N (s/\omega_0)^{2N}}$$

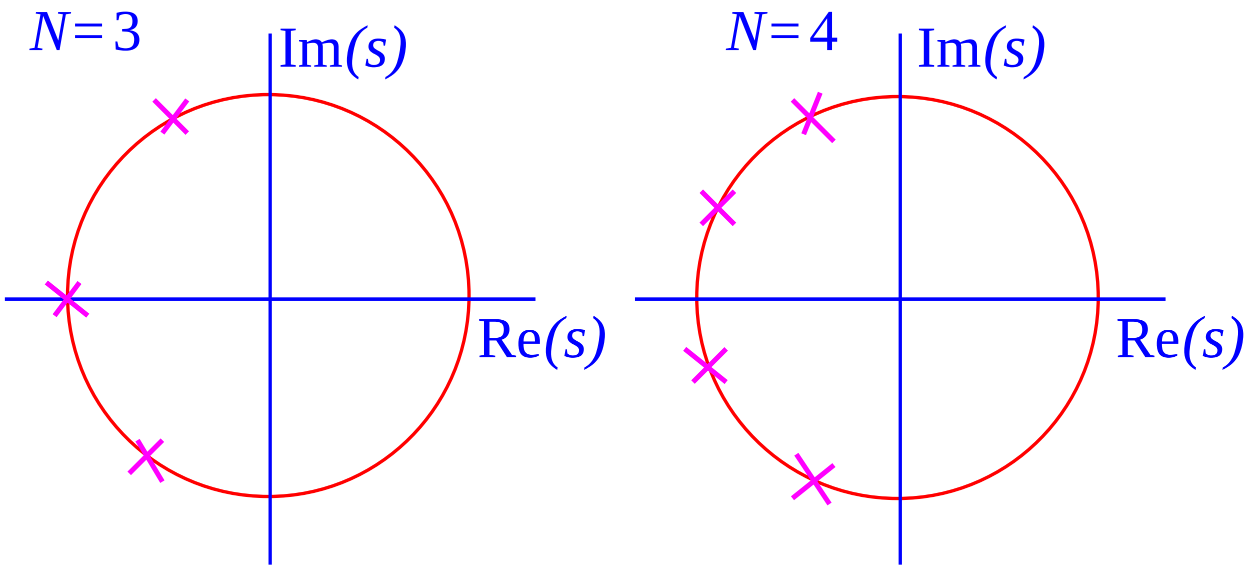
The $2N$ poles of this function:

$$s_k = \omega_0 \exp\left[j\frac{(N+1+2k)\pi}{2N}\right], \quad 0 \leq k \leq 2N-1$$

POLES OF $\left|H(s)\right|^2$



POLES OF $H(s)$



The Butterworth transfer function can be expressed as:

$$H(s) = \prod \frac{-s_k}{s - s_k}, \quad s = j\omega$$

$$s_k = \omega_0 \exp\left[j\left(\frac{\pi}{2} + \frac{(2k+1)\pi}{2N} \right) \right], \quad 0 \le k \le N-1$$

It is possible to determine ω_0, N, and $H(s)$ which correspond to $|H(\omega)|^2$ satisfying required specifications.

Example: $N = 1$, then $s_0 = \omega_0 \, e^{j\pi}$. Therefore

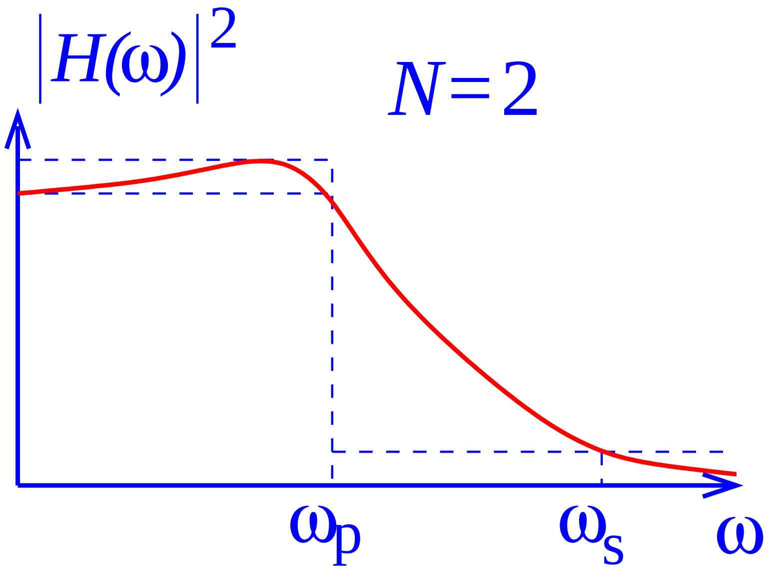
$$H(s) = \frac{1}{1 + s/\omega_0} \quad \Longrightarrow \quad |H(s)|^2 = \frac{1}{1 + (\omega/\omega_0)^2}$$

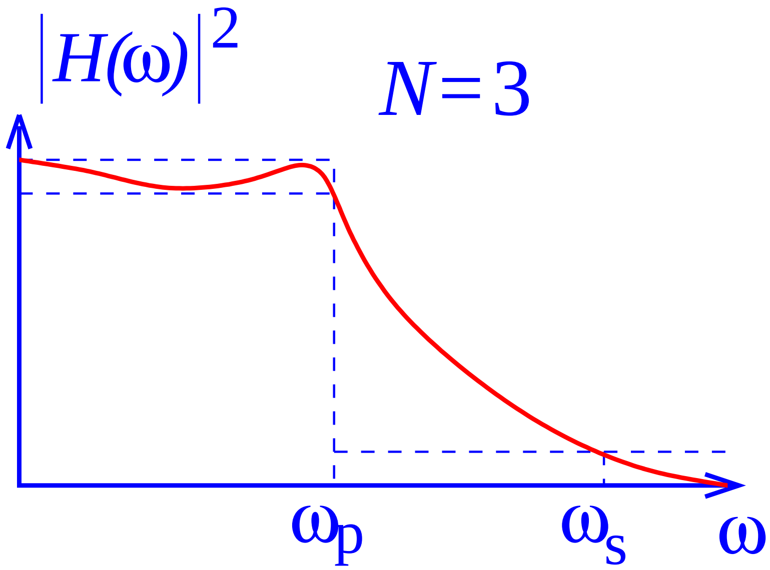
A lowpass Chebyshev filter is defined as:

$$|H(\omega)|^2 = \frac{1}{1 + \epsilon^2 T_N^2\left(\omega/\omega_0\right)}$$

where Chebyshev polynomial of degree N

$$T_N(x) = \begin{cases} \cos\left(N \arccos x\right), & |x| \leq 1 \\ \cosh\left(N \operatorname{arccosh} x\right), & |x| > 1 \end{cases}$$





Summary of analog filter design

- the s-plane poles and the transfer function of Chebyshev filter can be obtained in the way similar to that for Butterworth filter
- other analog filter structures are known, such as Bessel and elliptic filters
- an important question is what kind of transformation can be used to obtain digital IIR filters from the corresponding analog filters

We wish to find some transformations of $H(s)$ to $H(z)$, where $H(z)$ is a digital filter transfer function

Consider bilinear transformation which represents an algebraic unique mapping between the points in the s-plane and those in the z-plane. This transformation corresponds to replacing s by

$$\frac{2}{\Delta t} \cdot \frac{z-1}{z+1} = \frac{2}{\Delta t} \cdot \frac{1-z^{-1}}{1+z^{-1}}$$

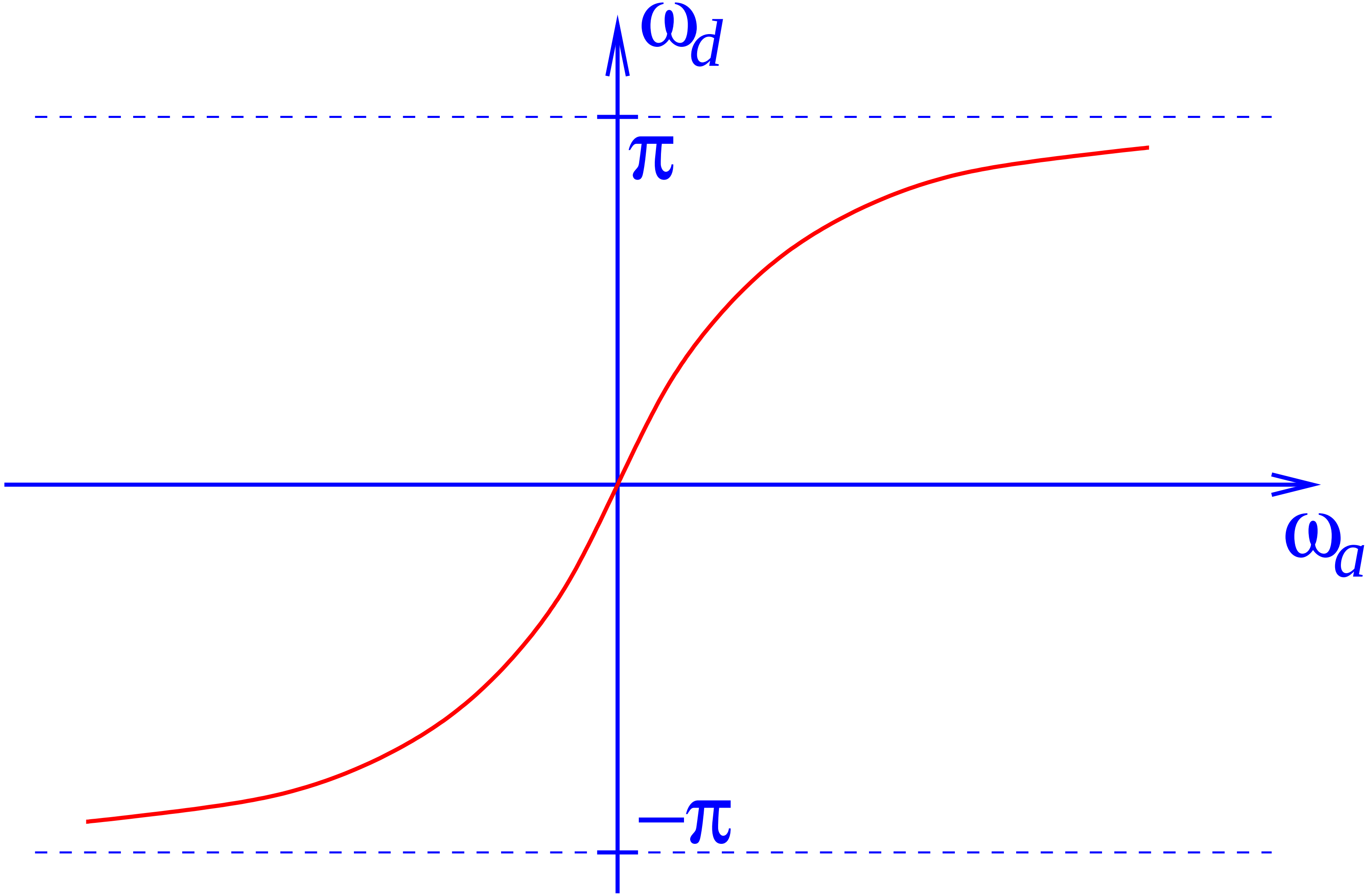
Substitute $s = j\omega_a$ (with analog frequency ω_a) and $z = \exp\{j\omega_d\}$ (with digital frequency ω_d) in the introduced bilinear transformation:

$$
\begin{aligned}
j\omega_a &= \frac{2}{\Delta t} \cdot \frac{\exp\{j\omega_d\} - 1}{\exp\{j\omega_d\} + 1} \\
&= \frac{2}{\Delta t} \cdot \frac{\exp\{j\omega_d/2\}\left(\exp\{j\omega_d/2\} - \exp\{-j\omega_d/2\}\right)}{\exp\{j\omega_d/2\}\left(\exp\{j\omega_d/2\} + \exp\{-j\omega_d/2\}\right)} \\
&= \frac{2j}{\Delta t} \tan\{\omega_d/2\}
\end{aligned}
$$

Hence, the bilinear transformation gives us the following mapping between analog and digital frequencies:

$$
\omega_a = \frac{2}{\Delta t} \tan\{\omega_d/2\} \qquad \omega_d = 2 \arctan\left\{\frac{\omega_a \Delta t}{2}\right\}
$$

$$\omega = 2\arctan(\omega_a \Delta t/2)$$



Bilinear transformation warps the digital frequency with respect to analog frequency. The nonlinear warping function is $2\,\arctan\{0.5\,\omega_a\,\Delta t\}$.

To design lowpass filter with the digital transition-region frequencies ω_P and ω_S, we find the analog prewarped frequencies

$$\tilde{\omega}_P = \frac{2}{\Delta t}\tan\{\omega_P/2\}\ , \quad \tilde{\omega}_S = \frac{2}{\Delta t}\tan\{\omega_S/2\} \qquad (*)$$

and design the analog filter using them in chosen specification. Then, the analog filter can be transformed to the digital one as:

$$H(z) = H(s)\Big|_{s=(2[z-1])/(\Delta t[z+1])} \qquad (**)$$

IIR filter design using bilinear transformation:

Step 1: Convert each specified edge-band (transition region) frequency of the desired digital filter to a corresponding edge-band frequency of an analog filter using $(*)$

Step 2: Design an analog filter $H(s)$ of the desired type, according to the transformed specifications

Step 3: Transform analog filter $H(s)$ to a digital filter $H(z)$ using $(**)$

Another method of IIR filter design does not require analog filter design and exploits the ARMA model

From LCCD equations, we know that the ARMA model reads

$$\sum_{k=0}^{N} a(k)y(n-k) = \sum_{k=0}^{M} b(k)x(n-k)$$

This model corresponds to the IIR filter with the frequency response

$$H(z) = \frac{\sum_{k=0}^{M} b(k)z^{-k}}{\sum_{k=0}^{N} a(k)z^{-k}}$$

Minimizing

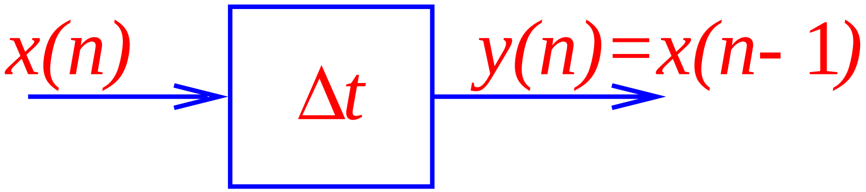
$$|\epsilon|^2 = \int_{-\pi}^{\pi} \left| H_{\mathrm{id}}(\omega) - \frac{\sum_{k=0}^{M} b(k) e^{-j\omega k}}{\sum_{k=0}^{N} a(k) e^{-j\omega k}} \right|^2 d\omega$$

we obtain desired filter parameters $a(k)$, $k = 1, \ldots, N$ and $b(k)$, $k = 1, \ldots, M$

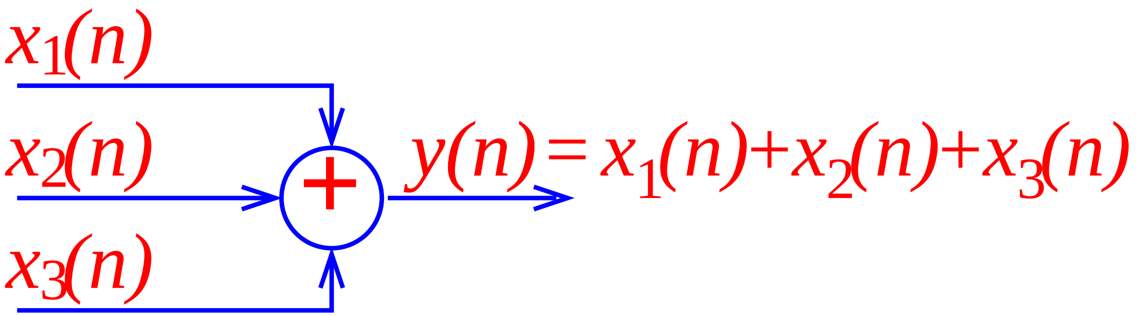
Efficient numerical optimization methods can be formulated to minimize this function

6.7 Realizations of digital filters

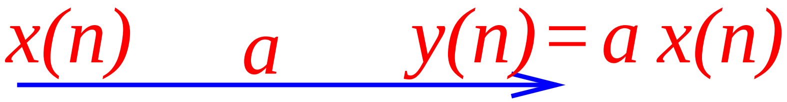
unit delay



adder/summator



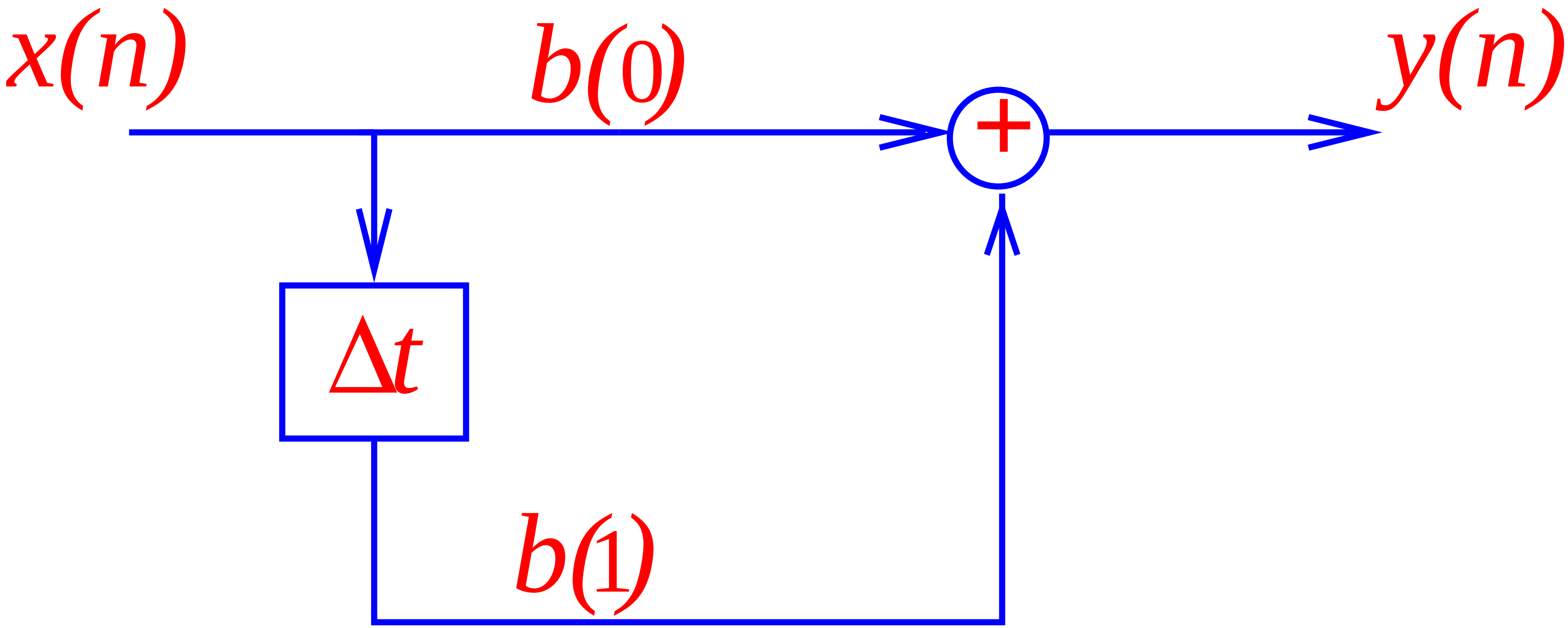
multiplier by a constant



BASIC BUILDING BLOCKS

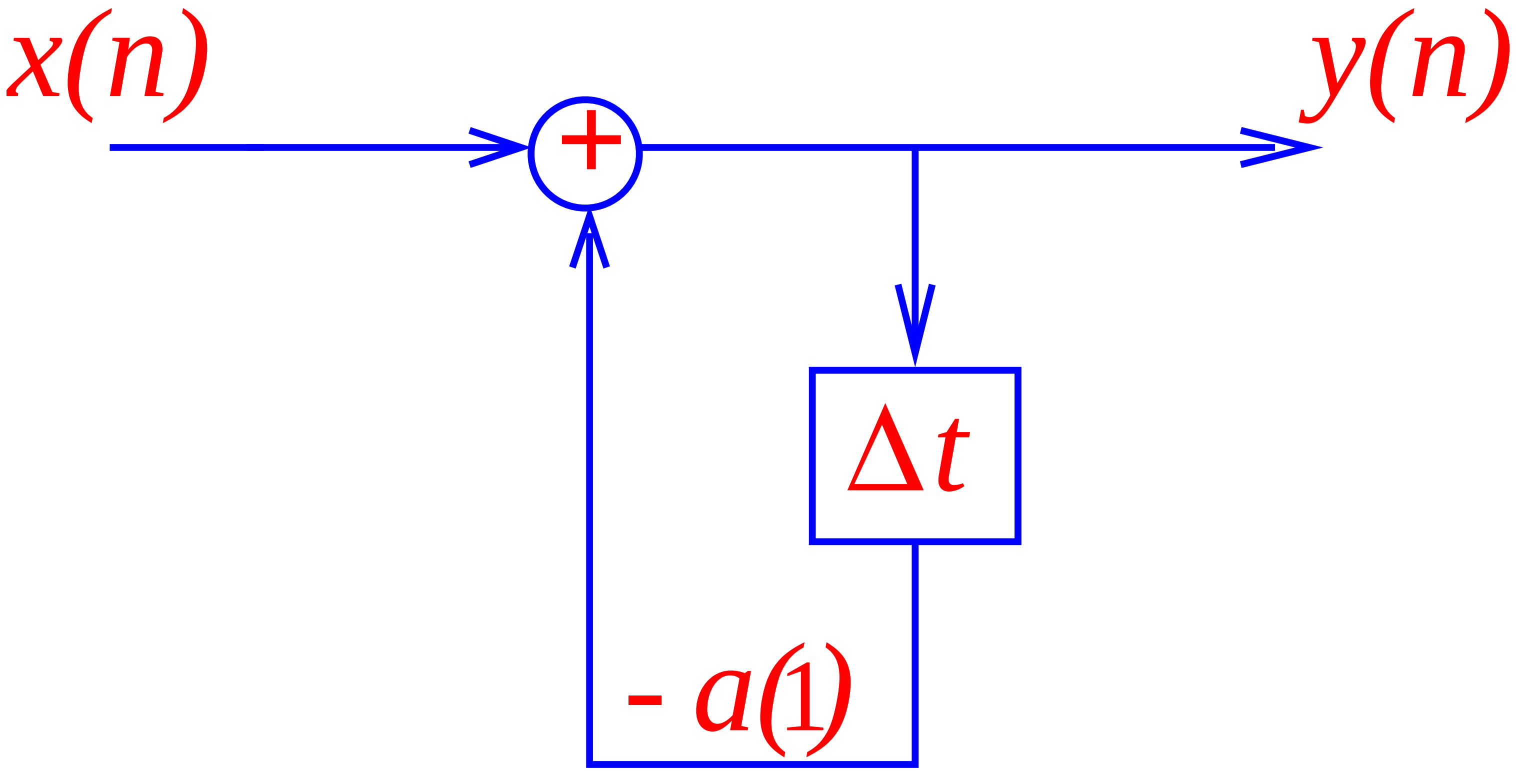
Example 1: First-order FIR filter

$$H(z) = b(0) + b(1)z^{-1}, \qquad y(n) = b(0)x(n) + b(1)x(n-1)$$



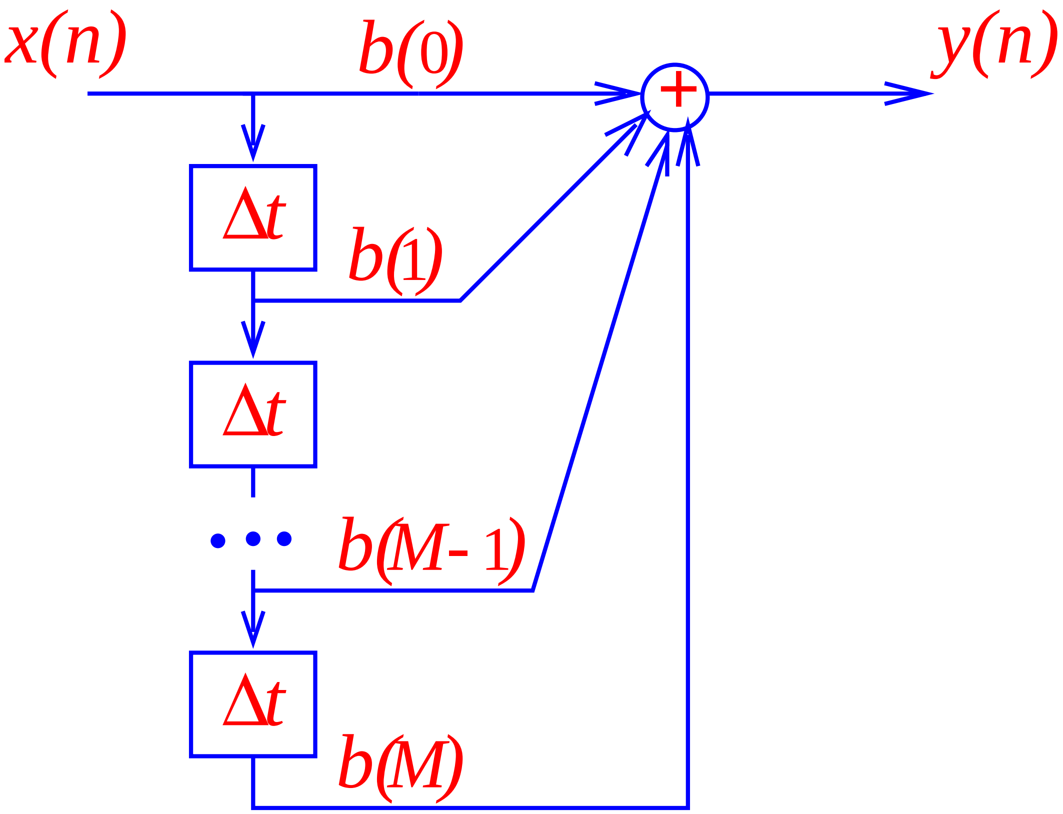
Example 1: First-order IIR filter

$$H(z) = \frac{1}{1 + a(1)z^{-1}}, \qquad y(n) = -a(1)y(n-1) + x(n)$$



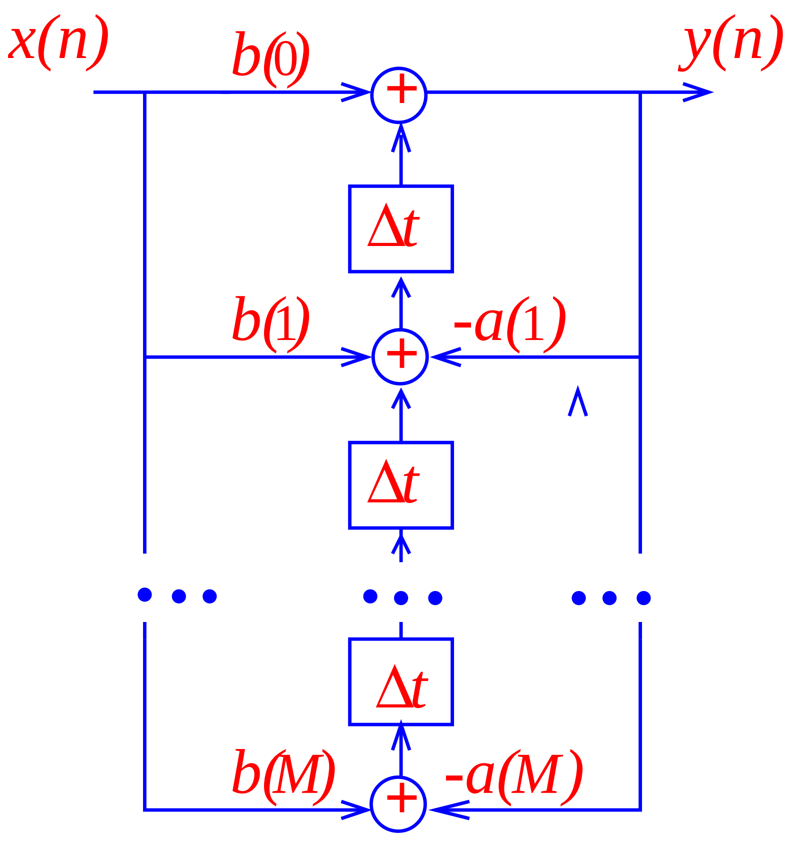
The general FIR filter case:

$$y(n) = \sum_{k=0}^{M} b(k)x(n-k)\,, \quad H(z) = \sum_{k=0}^{M} b(k)z^{-k}$$



The IIR filter case ($N = M$):

$$y(n) = \sum_{k=0}^{M} b(k)x(n-k) - \sum_{k=1}^{M} a(k)y(n-k), \quad H(z) = \frac{\sum_{k=0}^{M} b(k)z^{-k}}{\sum_{k=0}^{M} a(k)z^{-k}}$$



7 RANDOM SIGNAL ANALYSIS

7.1 Random Signals and Their Statistical Description

Let X be a random variable with the probability distribution

$$P_X(x) = \text{Probability}\{X \leq x\}$$

Probability density function (pdf):

$$p_X(x) = \frac{\partial P_X(x)}{\partial x}$$

where

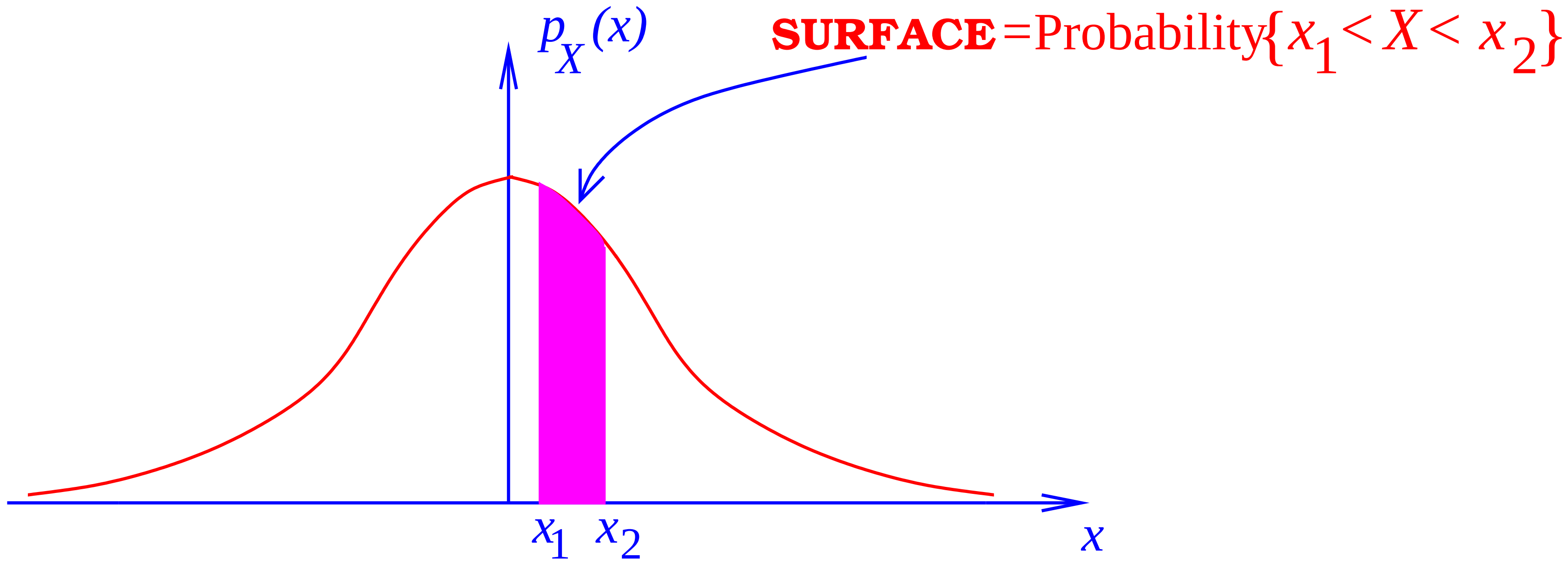
$$P_X(x_0) = \int_{-\infty}^{x_0} p_X(x)\, dx$$

Since $P_X(\infty) = 1$, we obtain

$$\int_{-\infty}^{\infty} p_X(x)\, dx = 1$$

Simple interpretation:

$$p_X(x) = \lim_{\Delta \to 0} \frac{\text{Probability}\{x - \Delta/2 \le X \le x + \Delta/2\}}{\Delta}$$



Mathematical expectation of an arbitrary function $f(X)$:

$$\mathrm{E}\{f(X)\} = \int_{-\infty}^{\infty} f(x)\, p_X(x)\, dx$$

Mean:

$$m_X = \mathrm{E}\{X\} = \int_{-\infty}^{\infty} x\, p_X(x)\, dx$$

Variance:

$$\mathrm{var}\{X\} = \sigma_X^2 = \mathrm{E}\{(X - \mathrm{E}\{X\})^2\} = \mathrm{E}\{X^2\} - \mathrm{E}\{X\}^2$$

Sample mean (recall MA systems!):

$$\hat{m}_X = \frac{1}{N} \sum_{n=0}^{N-1} x(n)$$

Sample variance:

$$\hat{\sigma}_X^2 = \frac{1}{N-1} \sum_{n=0}^{N-1} (x(n) - \hat{m}_X)^2$$

The weighting $1/N-1$ is used instead of $1/N$ to make the sample variance unbiased:

$$\mathrm{E}\{\hat{\sigma}_X^2\} = \sigma_X^2$$

For the complex random variable:

$$\begin{aligned} \mathrm{var}\{X\} = \sigma_X^2 &= \mathrm{E}\{(X - \mathrm{E}\{X\})^*(X - \mathrm{E}\{X\})\} \\ &= \mathrm{E}\{|X|^2\} - |\mathrm{E}\{X\}|^2 \end{aligned}$$

Random process (vector) $\mathbf{X} = (X(0), \ldots, X(N-1))^T$

$$\begin{aligned} P_{\mathbf{X}}(\mathbf{x}) &= \text{Probability}\{X(0) \le x(0) \text{ and } X(1) \le x(1) \\ &\quad \text{and } \cdots \text{ and } X(N-1) \le x(N-1)\} \end{aligned}$$

Probability density function (pdf):

$$p_{\mathbf{X}}(\mathbf{x}) = \frac{\partial^N P_{\mathbf{X}}(\mathbf{x})}{\partial x(0) \cdots \partial x(N-1)}$$

In the case of two random variables $\mathbf{X} = (x, y)^T$

$$p_{\mathbf{X}}(\mathbf{x}) = p_{XY}(x, y)$$

Variables are statistically independent when

$$p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y) \implies \mathrm{E}\{xy\} = \mathrm{E}\{x\}\,\mathrm{E}\{y\}$$

Expectation of a function $f(\mathbf{X})$:

$$\mathrm{E}\{f(\mathbf{X})\} \;=\; \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(\mathbf{x})\, p_{\mathbf{X}}(\mathbf{x})\, \mathbf{dx}$$

In the case of two random variables $\mathbf{X} = (x, y)^T$

$$\mathrm{E}\{f(X, Y)\} \;=\; \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, p_{X,Y}(x, y)\, dx\, dy$$

Correlation

$$r_{X,Y} = \mathrm{E}\{XY\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, y\; p_{X,Y}(x, y)\, dx\, dy$$

In the complex case

$$r_{X,Y} = \mathrm{E}\{XY^*\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, y^*\; p_{X,Y}(x, y)\, dx\, dy$$

Covariance matrix in the real case:

$$\mathbf{R} = \mathrm{E}\{\mathbf{X}\mathbf{X}^T\}$$

Covariance matrix in the complex case:

$$\mathbf{R} = \mathrm{E}\{\mathbf{X}\mathbf{X}^H\}$$

Structure of the covariance matrix:

$$\mathbf{R} = \begin{bmatrix} r(0) & r(1) & \cdots & r(N-1) \\ r^*(1) & r(0) & \cdots & r(N-2) \\ \cdots & \cdots & \cdots & \cdots \\ r^*(N-1) & r^*(N-2) & \cdots & r(0) \end{bmatrix}$$

where $r(i) = \mathrm{E}\{x(k)x^*(k+i)\}$ and the process is assumed to be in wide sense stationary

Result: Covariance matrix is Hermitian and nonnegative definite

To prove Hermitian property, we must show that $\mathbf{R}^H = \mathbf{R}$:

$$\mathbf{R}^H = \left(\mathrm{E}\{\mathbf{X}\mathbf{X}^H\}\right)^H = \mathrm{E}\{(\mathbf{X}\mathbf{X}^H)^H\} = \mathrm{E}\{\mathbf{X}\mathbf{X}^H\} = \mathbf{R}$$

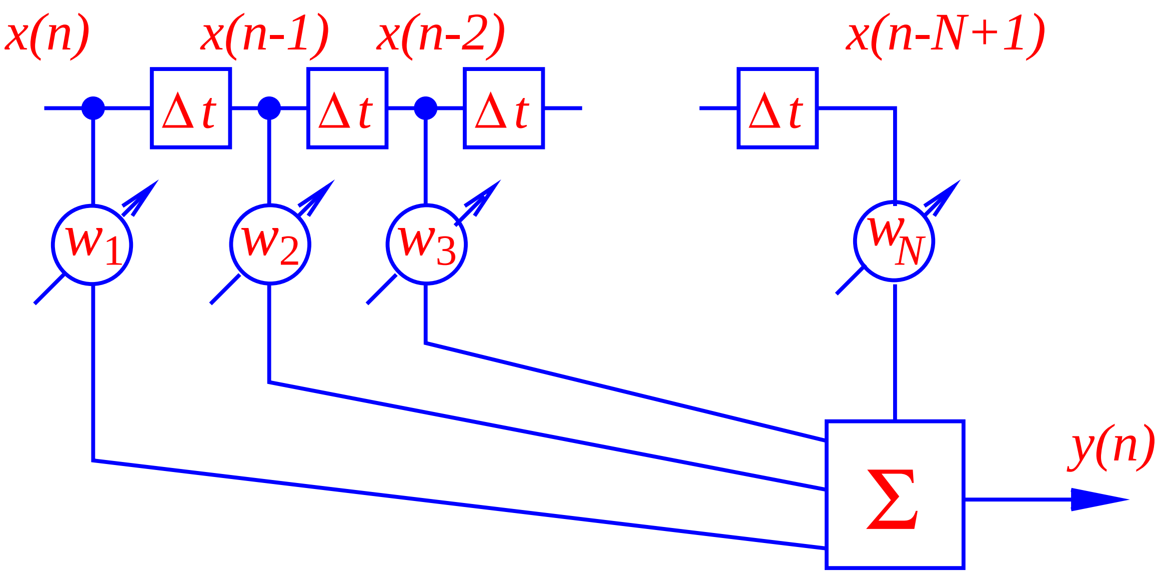
To prove nonnegative definiteness property we must show that

$\mathbf{d}^H\mathbf{R}\mathbf{d} \geq 0 \quad \forall \quad N \times 1 \;\; \text{vector} \;\; \mathbf{d}$

$$\mathbf{d}^H\mathbf{R}\mathbf{d} = \mathbf{d}^H\mathrm{E}\{\mathbf{X}\mathbf{X}^H\}\mathbf{d} = \mathrm{E}\{|\mathbf{d}^H\mathbf{X}|^2\} \geq 0$$

Corollary: Covariance matrix eigenvalues are real and nonnegative

7.2 Adaptive Filtering



The weight vector

$$\mathbf{W} = (w_1, w_2, \ldots, w_N)^T$$

is tuned according to some adaptation criterion

Observations vector

$$\begin{aligned} \mathbf{X}_k &= (x(k), x(k-1), \ldots, x(k-N+1))^T \\ &= \underbrace{s(k)\,\mathbf{a}_S}_{\text{desired signal}} + \underbrace{\mathbf{n}(k)}_{\text{noise and interference}} \end{aligned}$$

Complex filter output

$$y(k) = \mathbf{W}^H \mathbf{X}_k$$

The key idea of adaptive filtering is to maximize the desired signal and reject (as much as possible) everything else (noise, interference).

Assume that we know the frequency ω_S of the desired signal and let us maximize the Signal to Noise Ratio (SNR)

$$\mathrm{SNR} = \frac{|\mathbf{W}^H \mathbf{a}_S|^2}{\mathrm{E}\{|\mathbf{W}^H \mathbf{n}(n)|^2\}} = \frac{|\mathbf{W}^H \mathbf{a}_S|^2}{\mathbf{W}^H \mathbf{R_n} \mathbf{W}},$$

Noise (and interference) covariance matrix:

$$\mathbf{R_n} = \mathrm{E}\{\mathbf{n}(n)\mathbf{n}^H(n)\}$$

where $\mathrm{E}\{\cdot\}$ is statistical expectation

Known desired signal vector

$$\mathbf{a}_S = \left(1, e^{-j\omega_S}, \ldots, e^{-j(N-1)\omega_S}\right)^T$$

Characteristics of the interfering signals and noise are assumed unknown!

SNR maximization is equivalent to

$$\min_{\mathbf{W}} \mathbf{W}^H \mathbf{R_n} \mathbf{W} \quad \text{subject to} \quad \mathbf{W}^H \mathbf{a}_S = \text{const}$$

The (optimal) solution is

$$\mathbf{W}_{\text{opt}} = \alpha \mathbf{R}_{\mathbf{n}}^{-1} \mathbf{a}_S \qquad \longleftarrow \quad \text{Wiener filter}$$

(α does not affect SNR and, therefore, $\alpha = 1$ can be taken)

Optimal (maximally achievable) SNR can be obtained by inserting the Wiener solution into the SNR expression:

$$\text{SNR}_{\text{opt}} = \frac{|\mathbf{W}_{\text{opt}}^H \mathbf{a}_S|^2}{\mathbf{W}_{\text{opt}}^H \mathbf{R}_{\mathbf{n}} \mathbf{W}_{\text{opt}}} = \frac{(\mathbf{a}_S^H \mathbf{R}_{\mathbf{n}}^{-1} \mathbf{a}_S)^2}{\mathbf{a}_S^H \mathbf{R}_{\mathbf{n}}^{-1} \mathbf{a}_S} = \mathbf{a}_S^H \mathbf{R}_{\mathbf{n}}^{-1} \mathbf{a}_S$$

In practice, the matrix $\mathbf{R}_{\mathbf{n}}^{-1}$ is unavailable (the only available are the vectors $\mathbf{X}_k$)

Adaptive algorithm:

$$\mathbf{W}_{k+1} = \mathbf{W}_k + \mu(\mathbf{a}_S - (\mathbf{X}_k^H \mathbf{W}_k)\mathbf{X}_k)$$

Proof of convergence: Subtract $\mathbf{W}_{\text{opt}}$ from the both sides of the algorithm:

$$\mathbf{W}_{k+1} - \mathbf{W}_{\text{opt}} = \mathbf{W}_k - \mathbf{W}_{\text{opt}} + \mu \mathbf{a}_S - \mu \mathbf{X}_k \mathbf{X}_k^H (\mathbf{W}_k - \mathbf{W}_{\text{opt}} + \mathbf{W}_{\text{opt}})$$

Take expectation and remark that

$$\mathbf{V}_k = \mathrm{E}\{\mathbf{W}_k - \mathbf{W}_{\text{opt}}\}, \quad \mathbf{R_n} = \mathrm{E}\{\mathbf{X}_k \mathbf{X}_k^H\}$$

where we assume that the observation $\mathbf{X}_k$ does not contain the desired signal (this assumption can be relaxed later)

Then, the last expression can be rewritten as

$$\mathbf{V}_{k+1} = [\mathbf{I} - \mu \mathbf{R_n}]\, \mathbf{V}_k + \mu \underbrace{(\mathbf{a}_S - \mathbf{R_n W}_{\text{opt}})}_{=\mathbf{a}_S - \mathbf{R_n R_n}^{-1}\mathbf{a}_S = 0}$$

where we assume that the random vectors $\mathbf{X}_k$ and $\mathbf{W}_k$ are statistically independent (so that $\mathrm{E}\{\mathbf{X}_k \mathbf{X}_k^H \mathbf{W}_k\} = \mathbf{R}\, \mathrm{E}\{\mathbf{W}_k\}$).

Hence, we obtain the following iterative equation:

$$\mathbf{V}_{k+1} = [\mathbf{I} - \mu \mathbf{R_n}]\, \mathbf{V}_k$$

Sufficient condition of convergence:

$$||\mathbf{V}_{k+1}|| < ||\mathbf{V}_k||\,, \quad \forall k$$

Equivalently, the kth power of the matrix $\mathbf{I} - \mu \mathbf{R_n}$ must become zero as k becomes large. This is equivalent to the condition

$$0 < \mu < \frac{2}{\lambda_{\max}}$$

where $\lambda_{\max}$ is the maximal eigenvalue of $\mathbf{R_n}$.

In practice, a proper (stronger) condition is

$$0 < \mu < \frac{2}{\mathrm{trace}\{\mathbf{R_n}\}}$$

Remark: Sometimes, it is easier to represent the observations as

$$\mathbf{X}_k = (x(k), x(k+1), \ldots, x(k+N-1))^T$$

rather than in time-reversed order

$$\mathbf{X}_k = (x(k), x(k-1), \ldots, x(k-N+1))^T$$

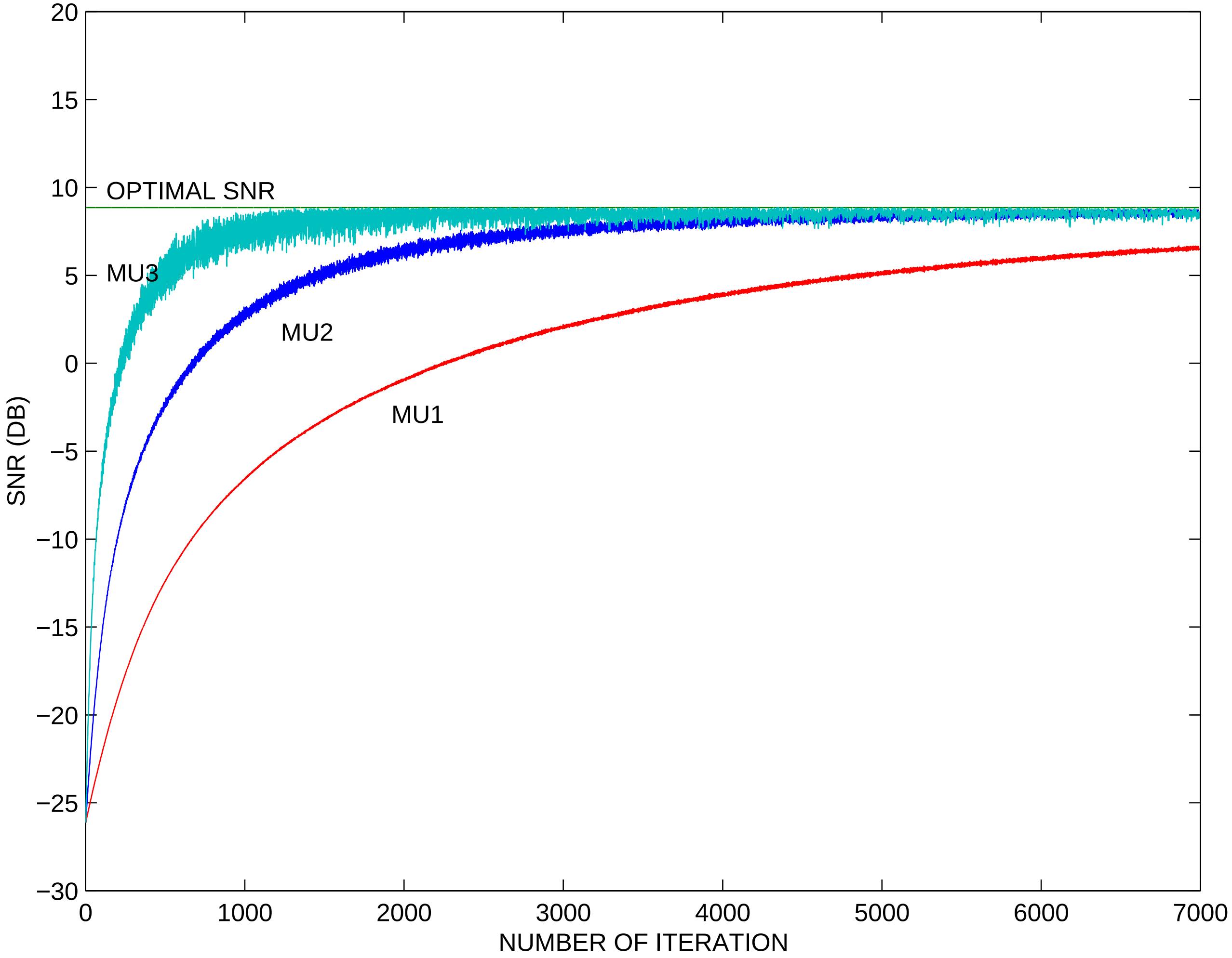
It is straightforward to reformulate the algorithm for this case. The definitions of $\mathbf{W}_k$ and $\mathbf{a}_S$ must be correspondingly modified.

Example: Let

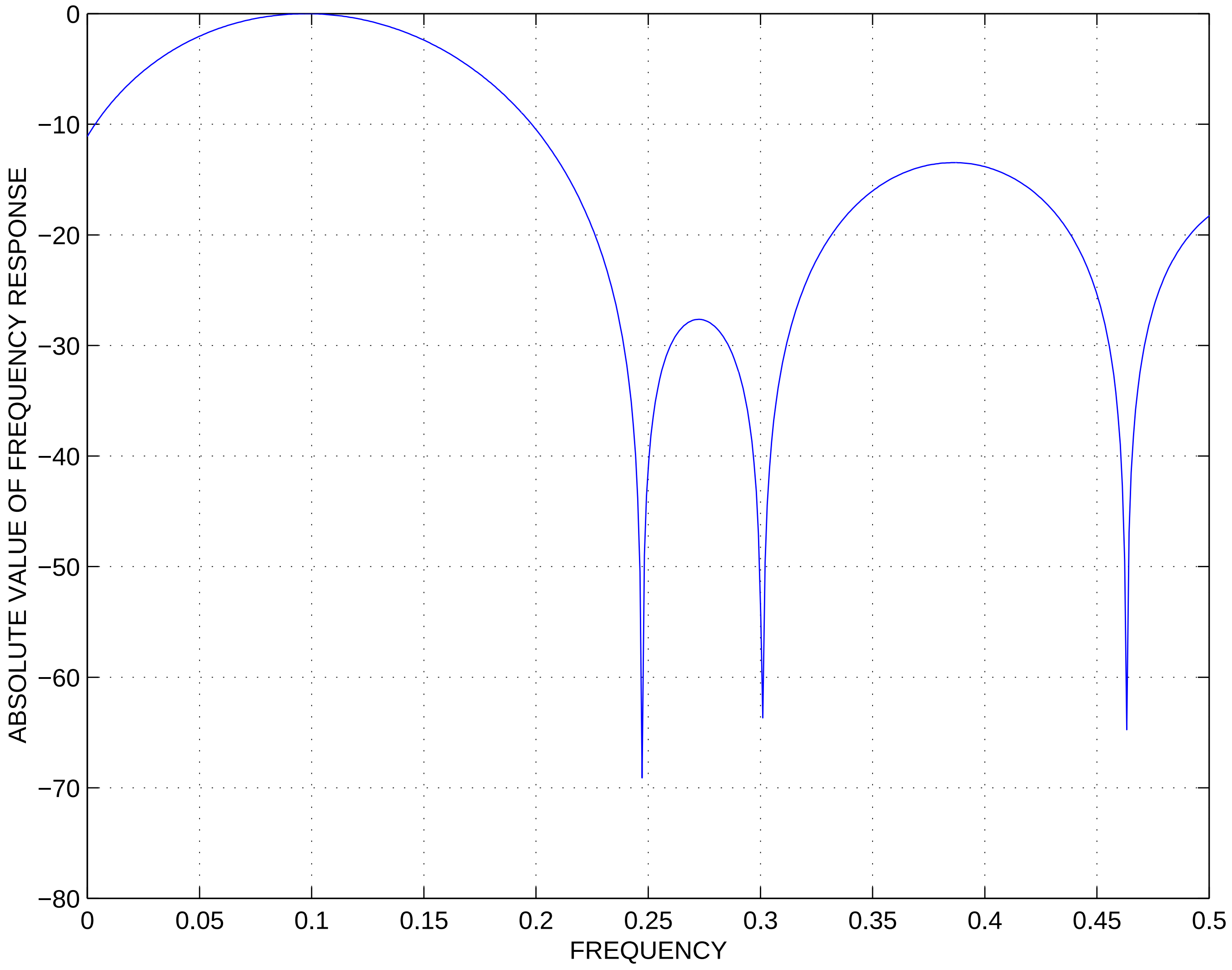
$$x(n) = \underbrace{\exp(j2\pi f_S n)}_{\text{desired signal}} + \underbrace{100\exp(j2\pi f_I n)}_{\text{interference}} + \underbrace{\xi(n)}_{\text{noise}}$$

Let us plot the so-called learning curve (SNR vs. k) for $N = 8$, $f_S = 0.1$, $f_S = 0.3$, and different values of μ

Comparison of convergence for $\mu_1 = 1/(50\,\mathrm{trace}\{\mathbf{R_n}\})$, $\mu_2 = 1/(15\,\mathrm{trace}\{\mathbf{R_n}\})$, and $\mu_3 = 1/(5\,\mathrm{trace}\{\mathbf{R_n}\})$



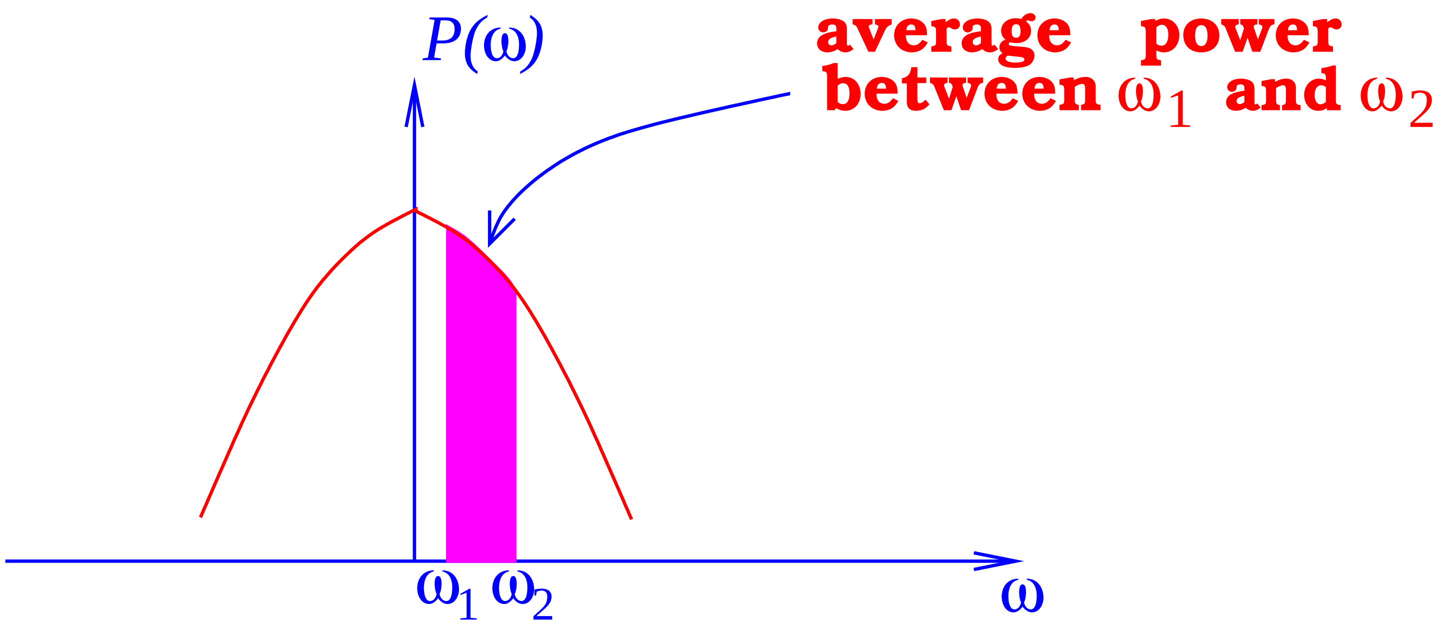
Frequency response in steady-state



7.3 Elements of Spectral Analysis

Definition of the power spectral density for a finite power random signal:

$$P(\omega) = \lim_{N\to\infty} \mathrm{E}\left\{ \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \right|^2 \right\}$$



Periodogram:

$$\hat{P}_p(\omega) = \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \right|^2$$

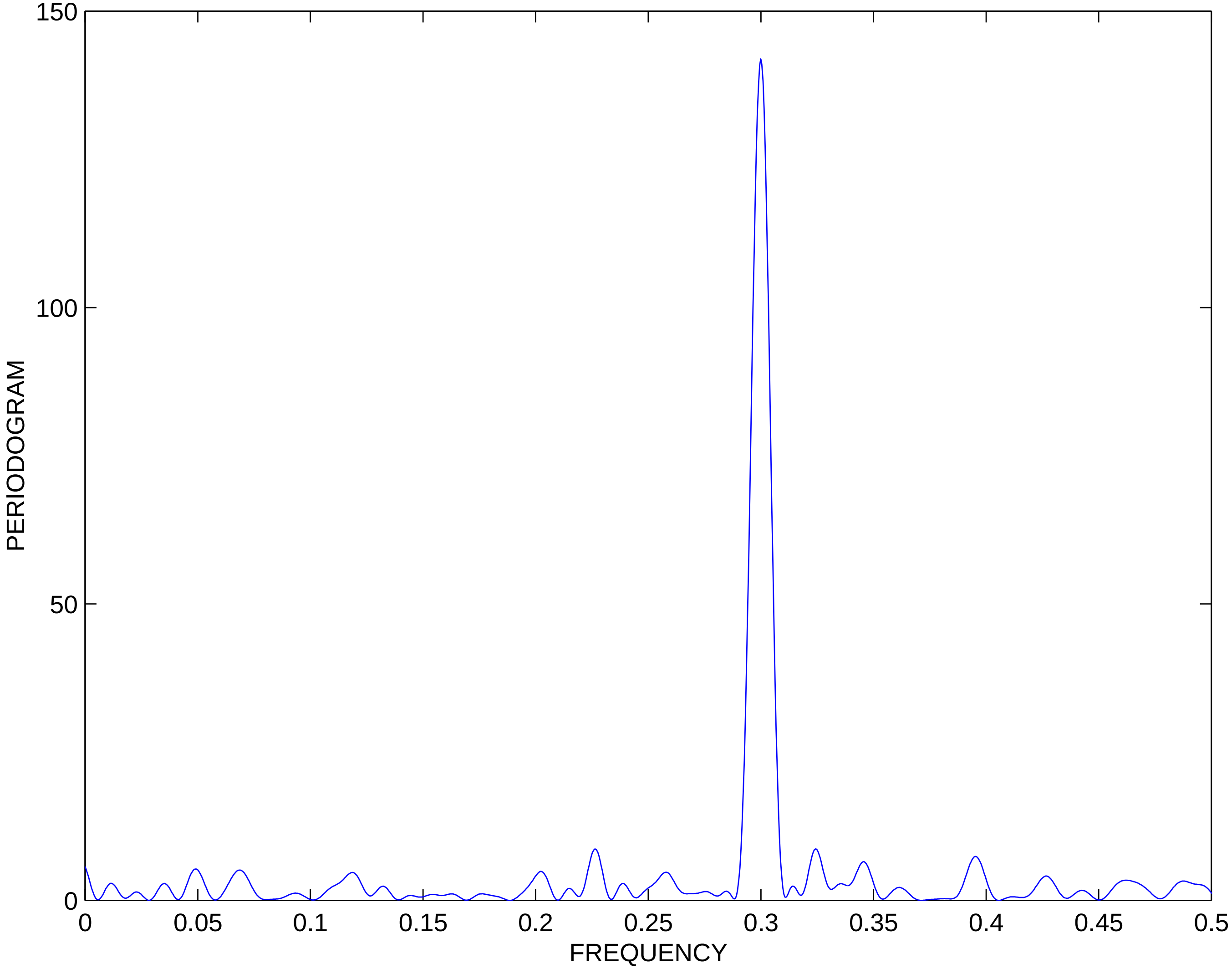
- periodogram is very similar to squared and normed DFT
- can be computed using FFT with zero-padding
- its variance does not reduce with N!

Example:

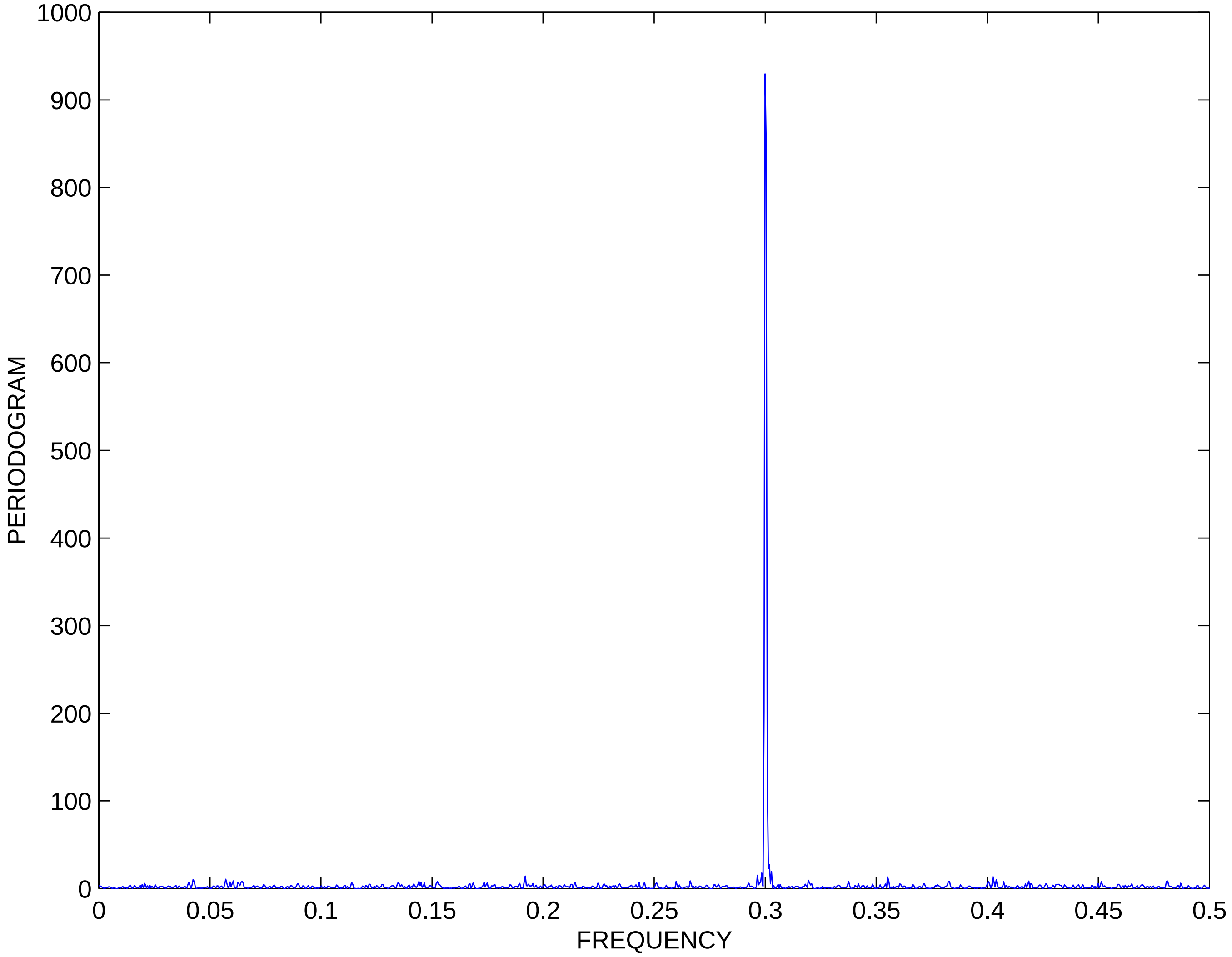
$$x(n) = A \exp(j2\pi f_S n) + \xi(n)$$

where $f_S = 0.3$ and ξ is zero-mean unit-variance complex noise.

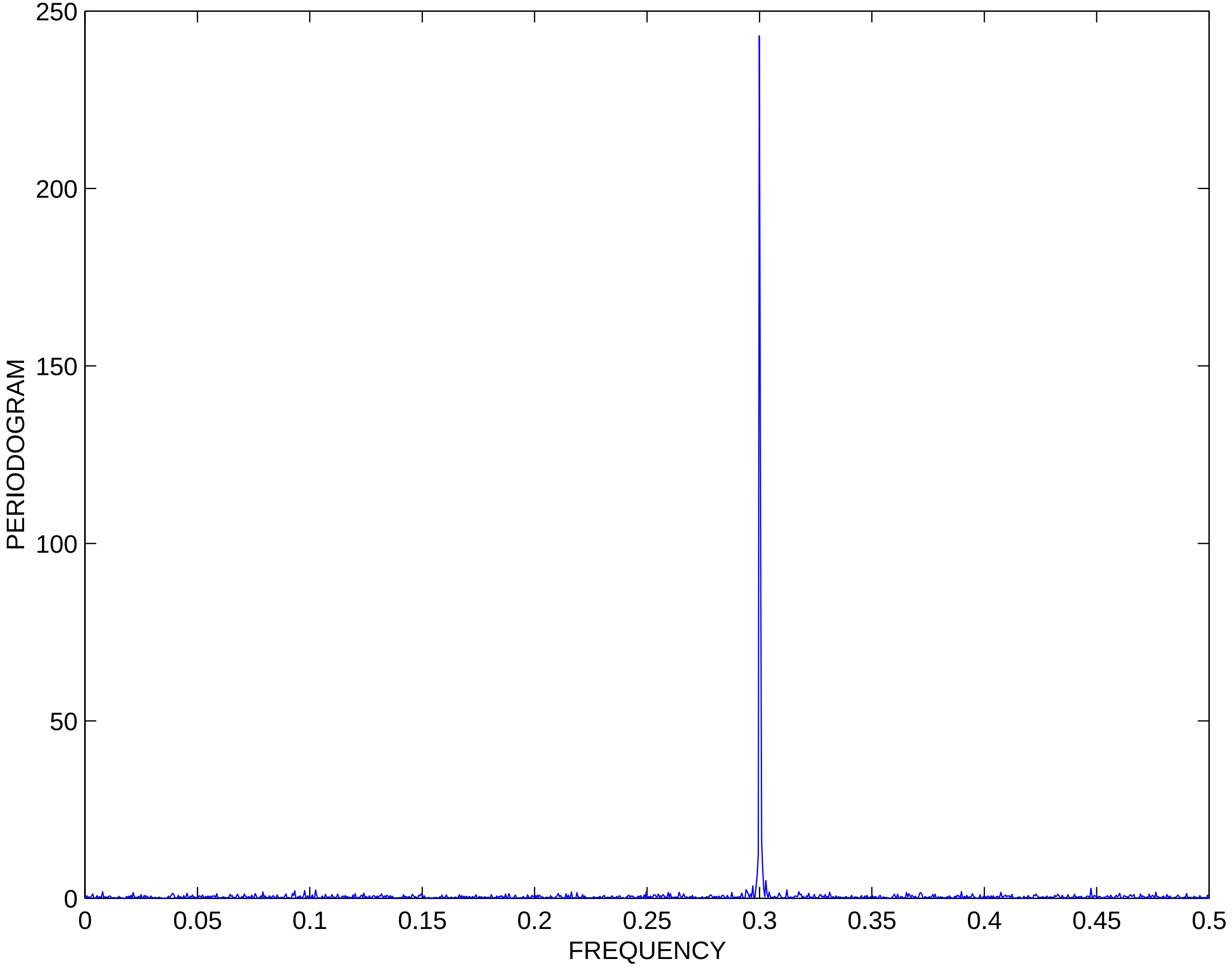
Periodogram $(A = 1,\ N = 100)$



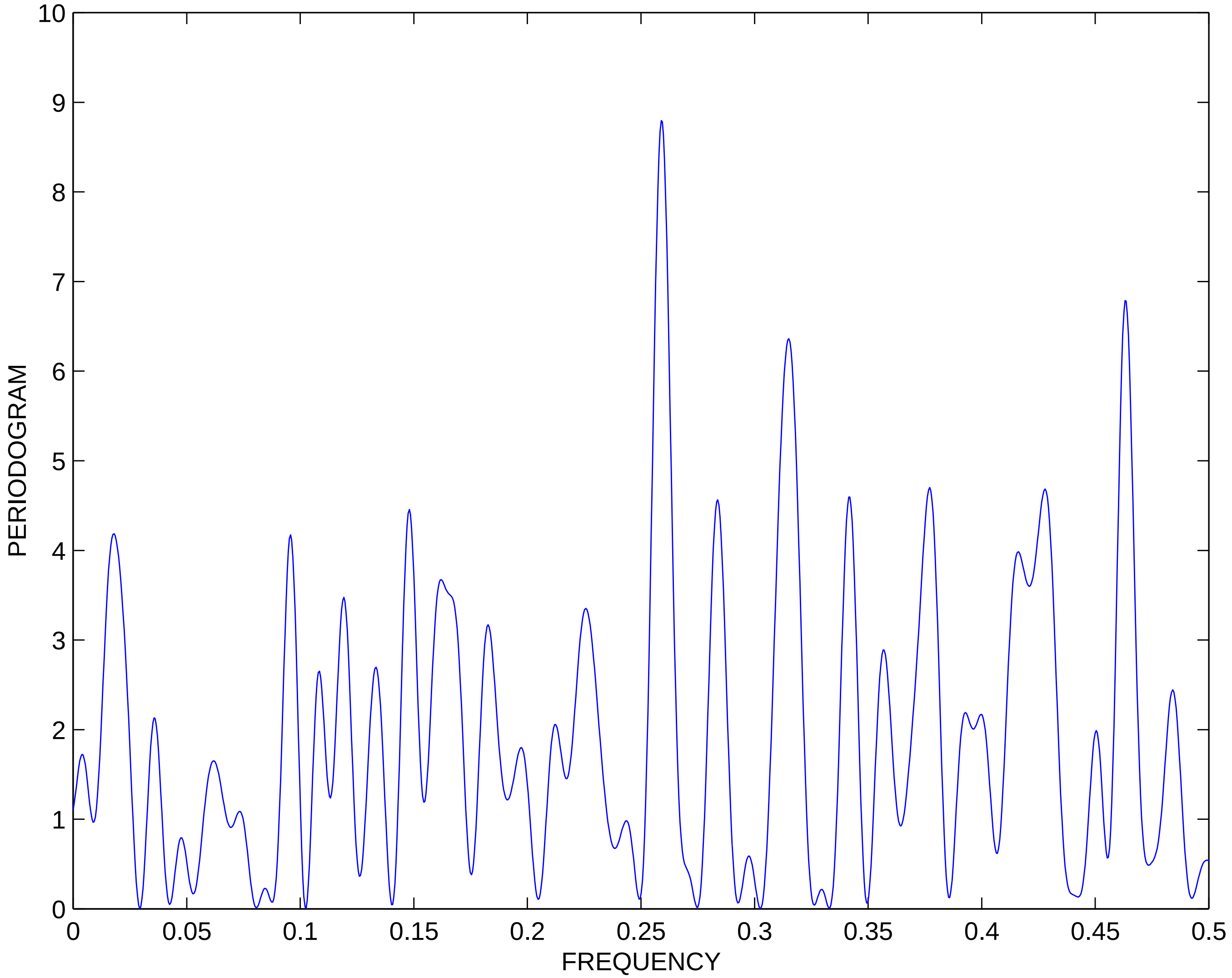
Periodogram ($A = 1$, $N = 1000$)



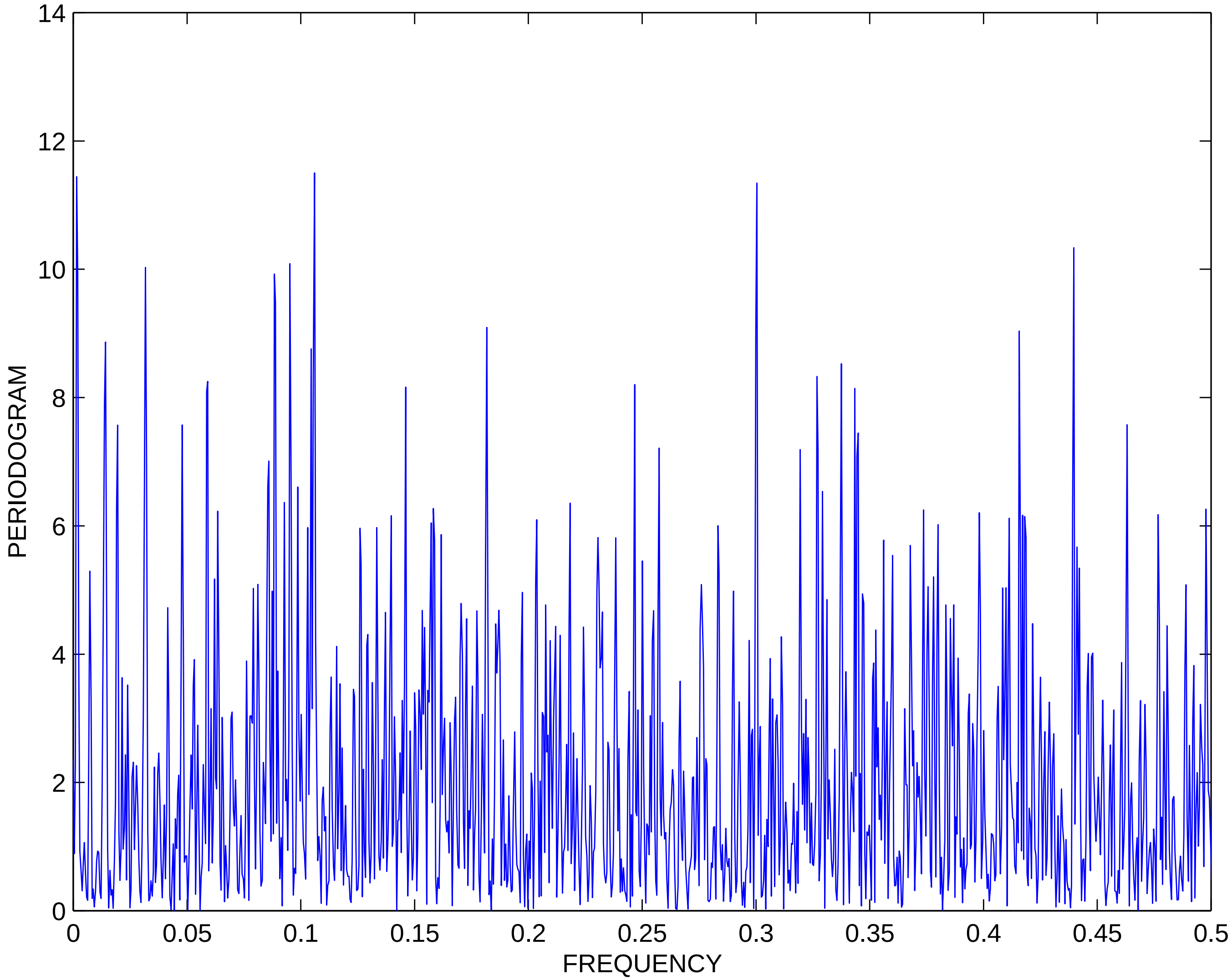
Periodogram ($A = 1,\ N = 10000$)



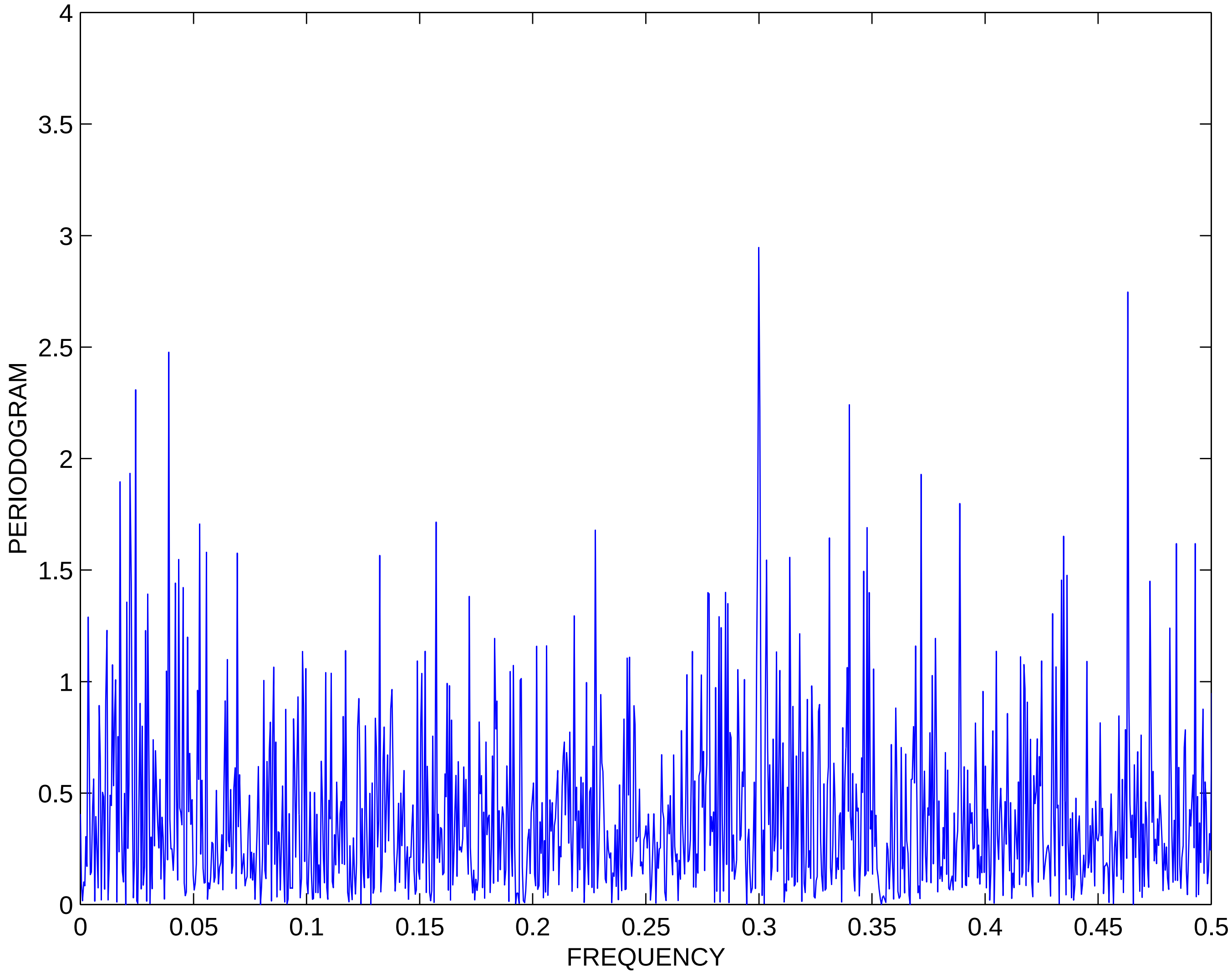
Periodogram ($A = 0.1$, $N = 100$)



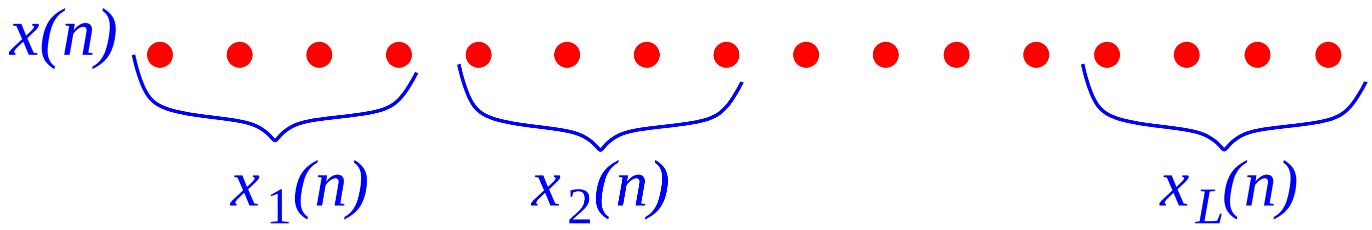
Periodogram ($A = 0.1$, $N = 1000$)



Periodogram $(A = 0.1,\ N = 10000)$



Refined periodogram (Bartlett method)

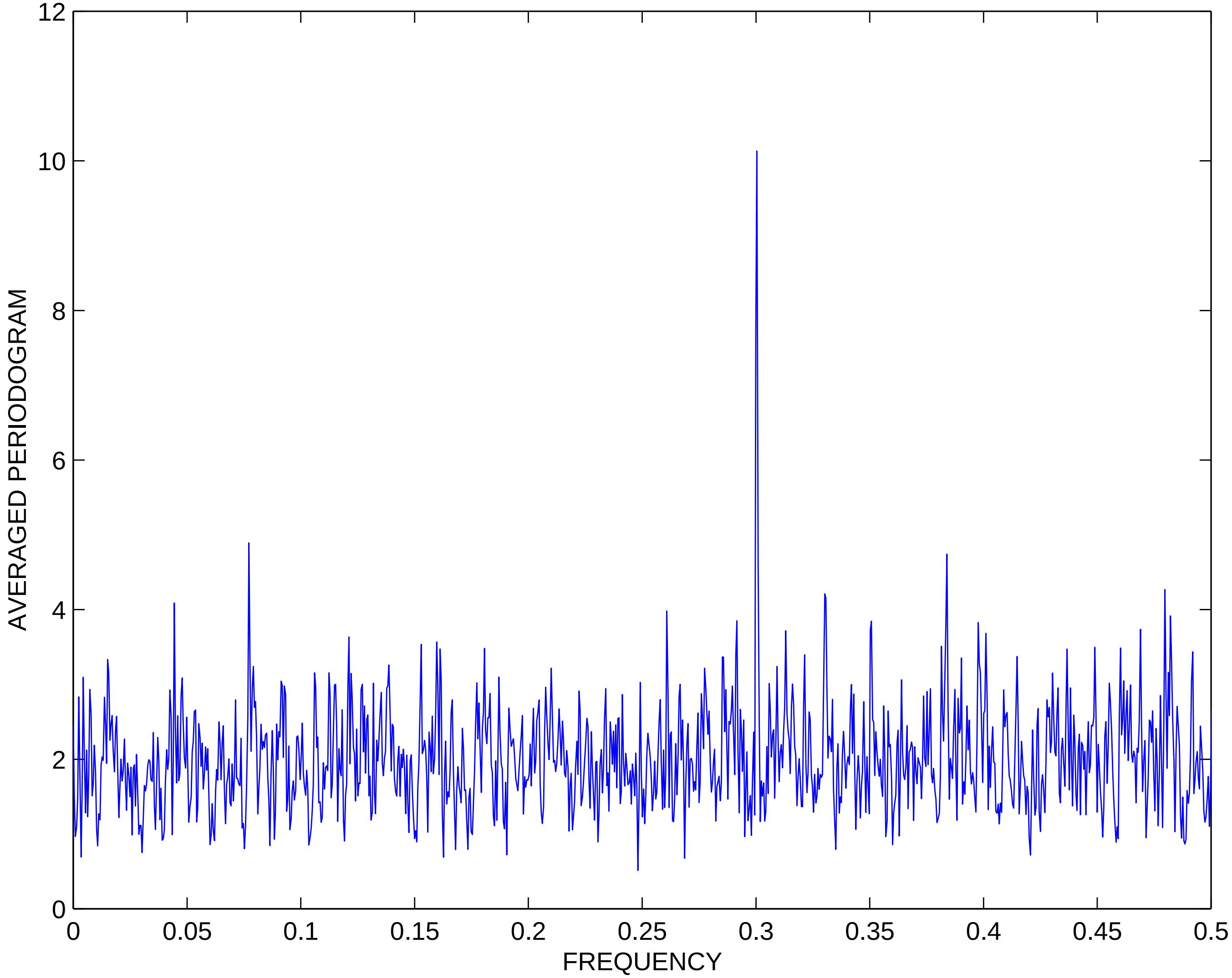


Based on dividing the original sequence into $L = N/M$ nonoverlapping subsequences of length M, computing periodogram for each subsequence, and averaging the result:

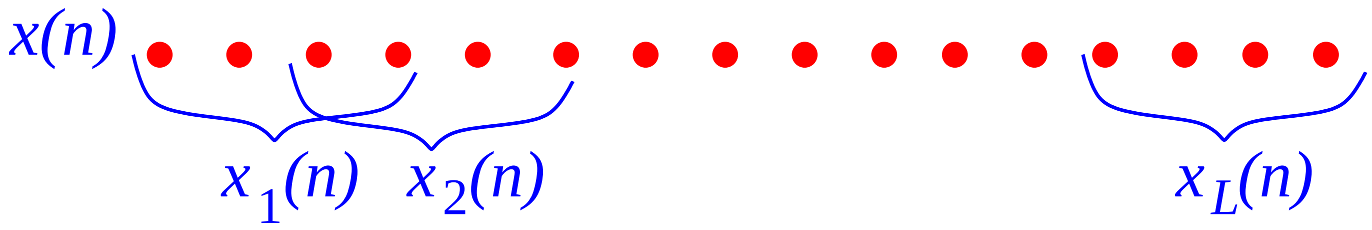
$$\hat{P}_B(\omega) = \frac{1}{L}\sum_{l=1}^{L}\hat{P}_l(\omega)\,,$$

$$\hat{P}_l(\omega) = \frac{1}{M}\left|\sum_{n=0}^{M-1} x_l(n)e^{-j\omega n}\right|^2$$

Averaged periodogram ($A = 0.1$, $N = 10000$, $M = 1000$)



Further refinements of periodogram (Welch method)

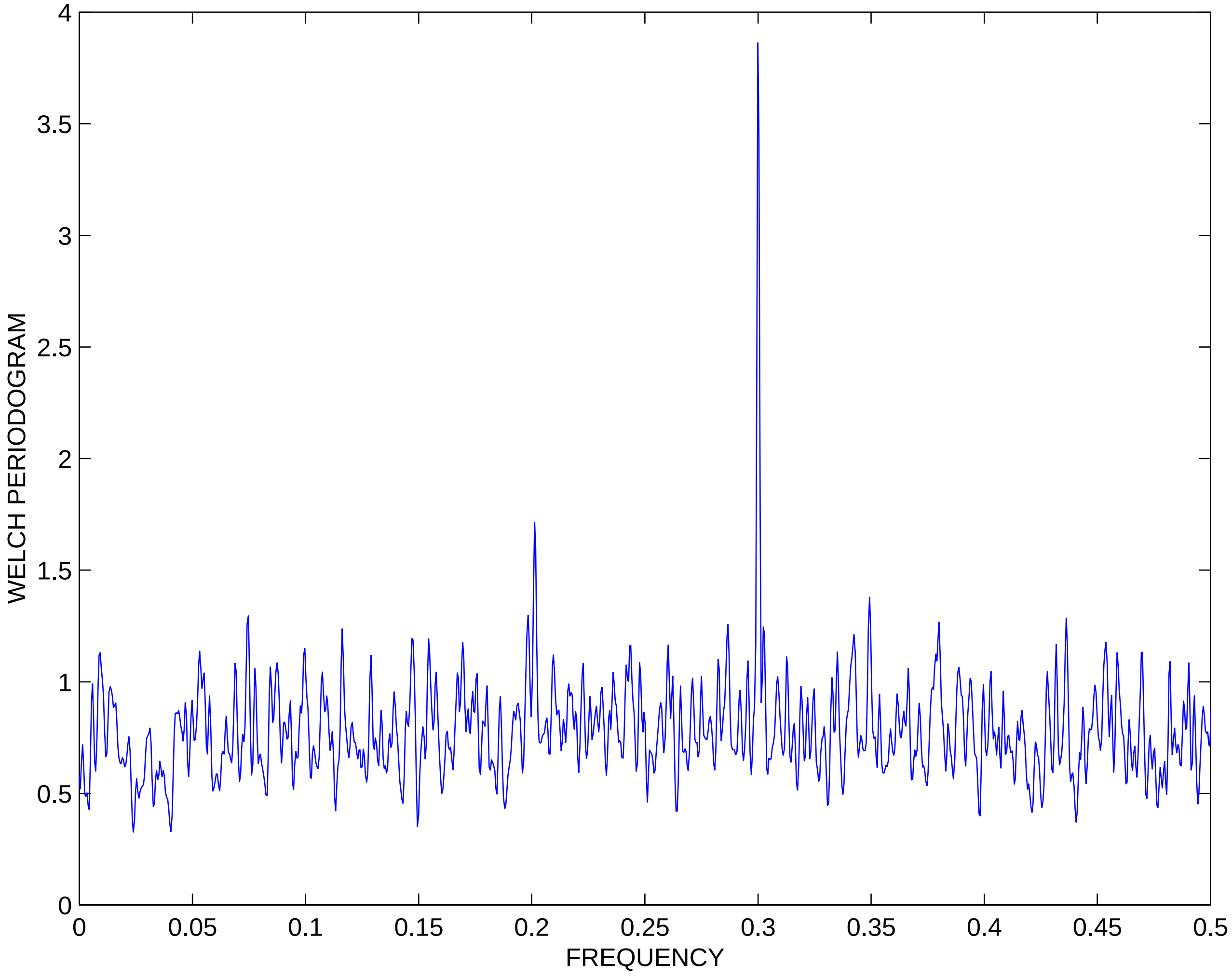


Welch method refines the Bartlett periodogram by:

- using overlapping subsequences
- windowing of each subsequence

Remark: There are many other advanced spectral analysis methods which perform significantly better than the periodogram methods

Welch periodogram ($A = 0.1$, $N = 10000$, $M = 1000$) with $2/3$ overlap and Hamming window



Another *alternative way* to define the power spectral density:

$$P(\omega) = \sum_{k=-\infty}^{\infty} r(k)\, e^{-j\omega k}$$

$$r(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\omega) e^{j\omega k}\, d\omega$$

where the *correlation function*

$$r(k) = \mathrm{E}\{x^*(n)x(n-k)\} = \mathrm{E}\{x(n)x^*(n+k)\}$$

Useful property:

$$r(-k) = r^*(k)$$

The two definitions of the power sprectral density are identical if $r(k)$ decays sufficiently fast.

Correlogram:

$$\hat{P}_c(\omega) = \sum_{k=-N+1}^{N-1} \hat{r}(k) e^{-j\omega k}$$

where we can use either *unbiased* or *biased* estimates of $r(k)$.

Unbiased estimate:

$$\hat{r}(k) = \begin{cases} \frac{1}{N-k} \sum_{i=k}^{N-1} x(i)^* x(i-k)\,, & k \geq 0 \\ \hat{r}^*(-k)\,, & k < 0 \end{cases}$$

Proof of the fact that it is unbiased:

$$\mathrm{E}\{\hat{r}(k)\} = \frac{1}{N-k} \sum_{i=k}^{N-1} \mathrm{E}\{x(i)^* x(i-k)\} = \frac{1}{N-k} \sum_{i=k}^{N-1} r(k) = r(k)$$

Example: Let the observation contains four points $\{x(0), x(1), x(2), x(3)\}$.

Unbiased estimate:

$$\hat{r}(0) = \frac{1}{4}[x(0)^*x(0) + x(1)^*x(1) + x(2)^*x(2) + x(3)^*x(3)]$$

$$\hat{r}(1) = \frac{1}{3}[x(1)^*x(0) + x(2)^*x(1) + x(3)^*x(2)]$$

$$\hat{r}(2) = \frac{1}{2}[x(2)^*x(0) + x(3)^*x(1)]$$

$$\hat{r}(3) = x(3)^*x(0)$$

Biased estimate:

$$\hat{r}(k) = \begin{cases} \frac{1}{N}\sum_{i=k}^{N-1} x(i)^* x(i-k)\,, & k \geq 0 \\ \hat{r}^*(-k)\,, & k < 0 \end{cases}$$

Prof of bias:

$$\mathrm{E}\{\hat{r}(k)\} = \frac{1}{N}\sum_{i=k}^{N-1} \mathrm{E}\{x(i)^* x(i-k)\} = \frac{1}{N}\sum_{i=k}^{N-1} r(k) = \frac{N-k}{N} r(k)$$

Biased estimate is *asymptotically* unbiased:

$$\lim_{N\to\infty} \frac{N-k}{N} r(k) = r(k)$$

Example (cont'd): four-point sequence.

Biased estimate:

$$\begin{aligned}
\hat{r}(0) &= \frac{1}{4}[x(0)^*x(0) + x(1)^*x(1) + x(2)^*x(2) + x(3)^*x(3)] \\
\hat{r}(1) &= \frac{1}{4}[x(1)^*x(0) + x(2)^*x(1) + x(3)^*x(2)] \\
\hat{r}(2) &= \frac{1}{4}[x(2)^*x(0) + x(3)^*x(1)] \\
\hat{r}(3) &= \frac{1}{4}x(3)^*x(0)
\end{aligned}$$

Hence, least reliable estimates are weighted by lowest weights!

Advantage of biased estimate: it is *more reliable* than the unbiased one because it assigns lower weights to the poorer estimates of long correlation lags.

Is there any relationship between the periodogram and correlogram?

Result: Correlogram computed through the biased estimate of $r(k)$ coincides with periodogram.

Proof: Consider the *auxiliary* signal

$$y(m) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x(k)\epsilon(m-k)$$

where $\{x(k)\}$ are fixed constants and $\{\epsilon(k)\}$ is a unit-variance white noise:

$$r_\epsilon(m-l) = \mathrm{E}\{\epsilon^*(m)\epsilon(l)\} = \delta(m-l)$$

The signal $y(m)$ can be viewed as the output of the filter with the transfer function

$$X(\omega) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x(k) e^{-j\omega k}$$

Relationship between filter *input* and *output* PSD's:

$$\begin{aligned}
P_y(\omega) &= |X(\omega)|^2 P_\epsilon(\omega) = |X(\omega)|^2 \sum_{k=-\infty}^{\infty} r_\epsilon(k) e^{-j\omega k} \\
&= |X(\omega)|^2 \sum_{k=-\infty}^{\infty} \delta(k) e^{-j\omega k} = |X(\omega)|^2 \\
&= \frac{1}{N} \left| \sum_{n=0}^{N-1} x(n) e^{-j\omega n} \right|^2 = \hat{P}_p(\omega)
\end{aligned}$$

Now, it remains to prove that $P_y(\omega) = \hat{P}_c(\omega)$. Remark that

$$
\begin{aligned}
r_y(k) &= \mathrm{E}\{y(m)^* y(m-k)\} \\
&= \frac{1}{N}\mathrm{E}\left\{\left[\sum_{p=0}^{N-1} x^*(p)\epsilon^*(m-p)\right]\left[\sum_{s=0}^{N-1} x(s)\epsilon(m-k-s)\right]\right\} \\
&= \frac{1}{N}\sum_{p=0}^{N-1}\sum_{s=0}^{N-1} x^*(p)x(s)\,\mathrm{E}\{\epsilon^*(m-p)\epsilon(m-k-s)\} \\
&= \frac{1}{N}\sum_{p=0}^{N-1}\sum_{s=0}^{N-1} x^*(p)x(s)\,\delta(p-k-s) \\
&= \frac{1}{N}\sum_{p=k}^{N-1} x(p)^* x(p-k) = \begin{cases} \hat{r}_x(k)\,, & 0 \le k \le N-1 \\ 0\,, & k \ge N \end{cases} \qquad \text{biased}
\end{aligned}
$$

Inserting the last result in the first definition of PSD, we obtain

$$\begin{aligned} P_y(\omega) &= \sum_{k=-\infty}^{\infty} r_y(k)e^{-j\omega k} \\ &= \sum_{k=-N+1}^{N-1} \hat{r}_x(k)e^{-j\omega k} = \hat{P}_c(\omega) \end{aligned}$$

Summarizing, we have proven the following two equalities for the spectrum of the auxiliary signal $y(n)$:

$$P_y(\omega) = \hat{P}_p(\omega)\,, \qquad P_y(\omega) = \hat{P}_c(\omega) \qquad \Longrightarrow$$

$$\hat{P}_p(\omega) = \hat{P}_c(\omega)$$

Refined correlogram (Blackman-Tukey method):

- $\hat{r}(k)$ is a *poor* estimate for higher lags k. Hence, let us truncate it (use $M \ll N$ points)
- Let us use some *lag window*

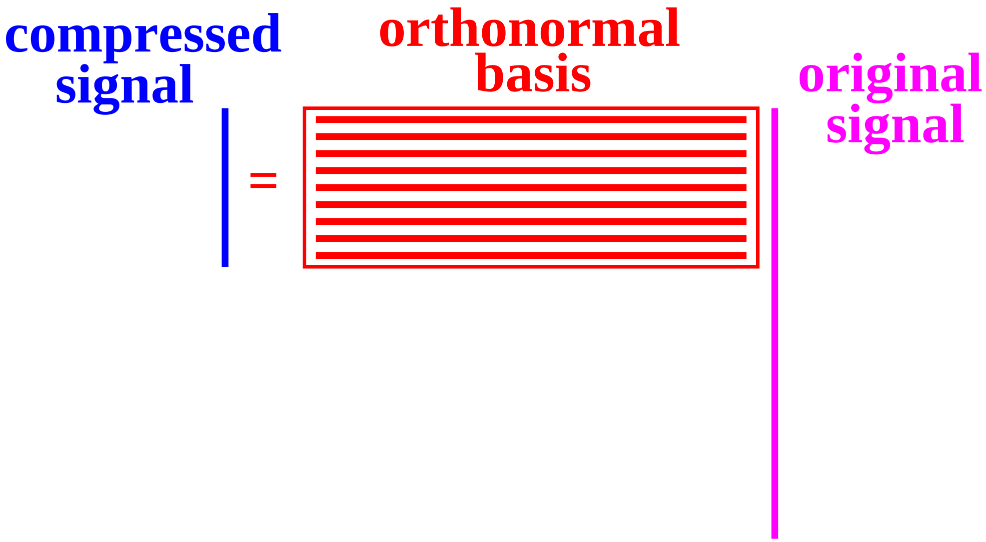
$$\hat{P}_{BT}(\omega) = \sum_{k=-M+1}^{M-1} w(k)\hat{r}(k)e^{-j\omega k}$$

7.4 Signal and Image Compression

Definition: Compression is the operation of representing the N-bit information using N_c bits.

$$\text{compression ratio} = \frac{N}{N_c}$$

Principle of compression by orthonormal transforms:



Original signal:

$$\mathbf{x} = (x(0), x(1), \ldots, x(M-1))^T$$

Compression:

$$\mathbf{X} = (X(0), X(1), \ldots X(L-1))^T = \mathbf{T}^H \mathbf{x}$$

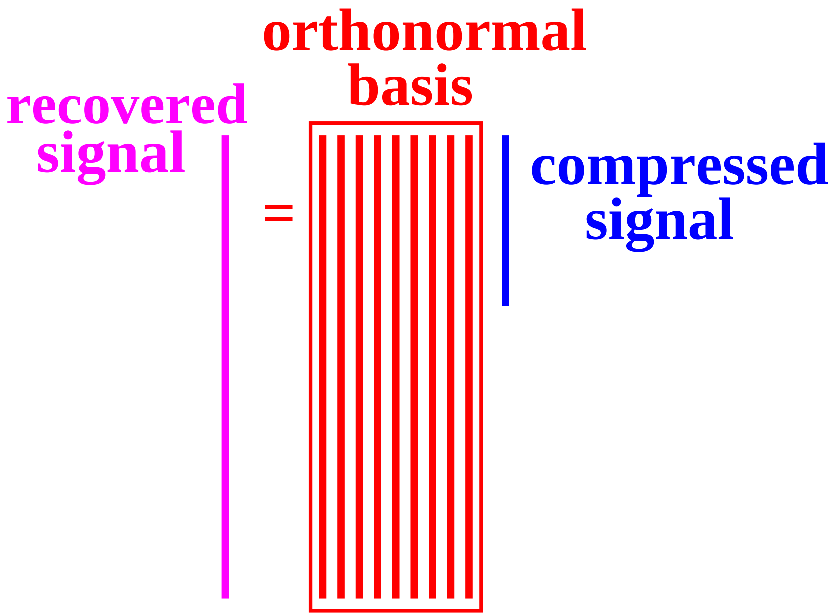
where

$$\frac{M}{L} = \frac{N}{N_c}$$

and $\mathbf{T}$ is the $M \times L$ transformation matrix. The signal $\mathbf{X}$ can be interpreted as a vector of coefficients of expansion using orthonormal basis $\mathbf{T}$:

$$\tilde{\mathbf{x}} = \mathbf{T}\mathbf{X}$$

Signal reconstruction:



Orthonormal transform compression is lossy!

Good candidates for the transform $\mathbf{X}$:

- DFT
- DCT
- Karhunen-Loeve (principal eigenvectors based) transform

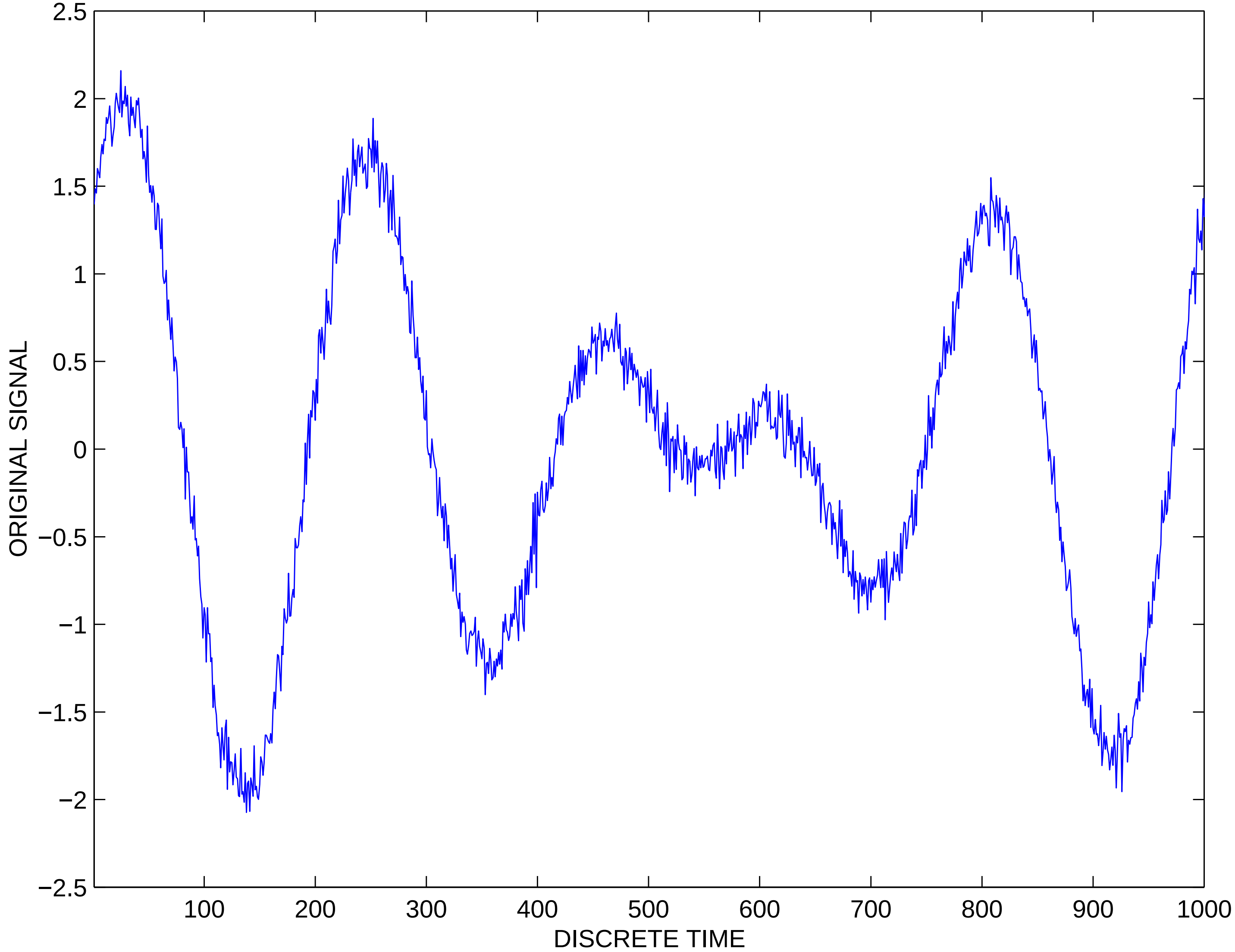
SVD of a $N \times N$ matrix

$$\mathbf{R} = \sum_{i=1}^{N} \lambda_i \mathbf{U}_i \mathbf{V}_i^H \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$$

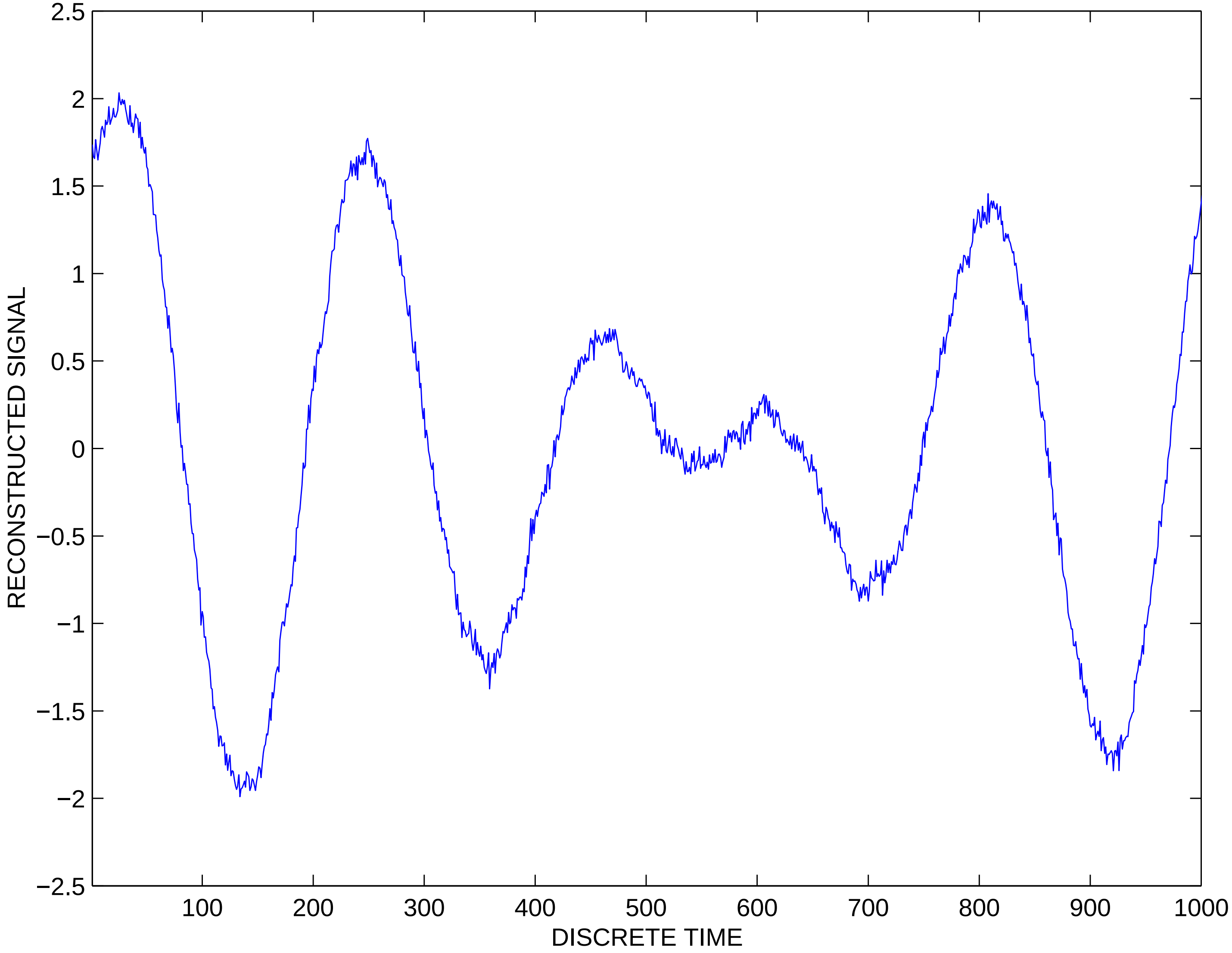
Compressed representation

$$\mathbf{R} \simeq \sum_{i=1}^{M} \lambda_i \mathbf{U}_i \mathbf{V}_i^H , \quad M < N$$

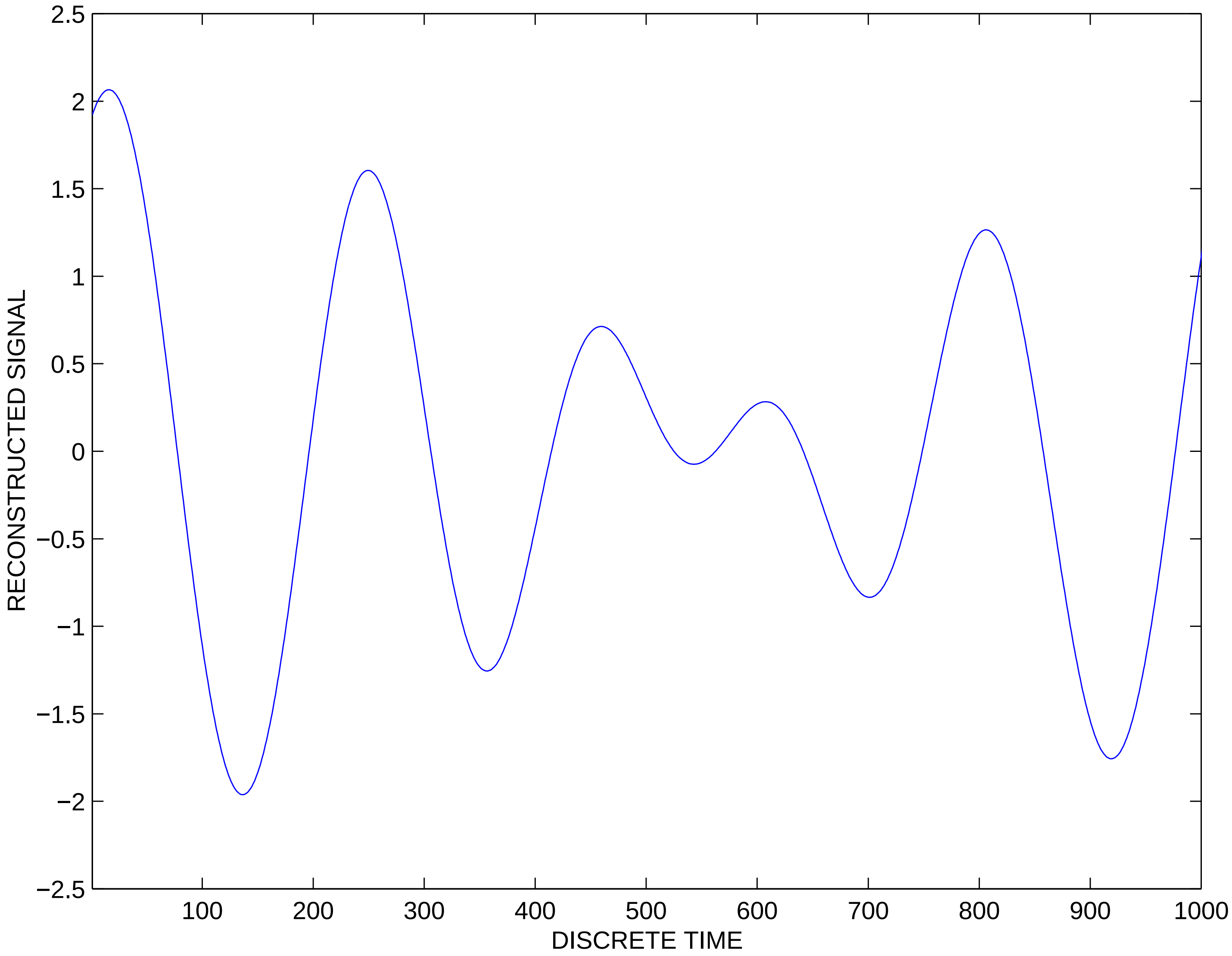
Example: the original (noisy) signal



Reconstructed signal after compression with the compression ratio $= 10$



Reconstructed signal after compression with the compression ratio $= 100$



CONCLUSIONS

- many of you will work on DSP problems in future
- when working with digital signals, do not forget their sampling rate!
- we considered only the most general topics of DSP
- to broaden and deepen your knowledge in specific DSP topics, an advanced DSP course might be needed
- DSP makes fun!