

CHAPTER 10 - FFT

10.1

Radix 2 4-point FFT

The 4-point DFT equation may be written as:

$$X(k) = \sum_{n=0}^3 x(n) W_4^{nk}$$

Following equations (10.3)-(10.5) of chapter 10 in the textbook, this may be split into even and odd sums as follows:

$$X(k) = \sum_{m=0}^1 x(2m)W_4^{2mk} + \sum_{m=0}^1 x(2m+1)W_4^{(2m+1)k}$$

The first term on the right hand side corresponds to even values of n ($x(0)$, $x(2)$) and the second to odd values ($x(1)$, $x(3)$). Taking out the constant twiddle factor from the second term on the right hand side gives:

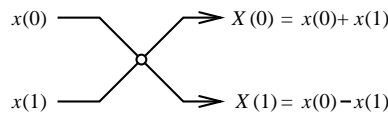
$$X(k) = \sum_{m=0}^1 x(2m)W_2^{mk} + W_4^k \sum_{m=0}^1 x(2m+1)W_2^{mk} = X_1(k) + W_4^k X_2(k) \quad (1)$$

We recognise that $X(k)$ is the sum of two 2-point DFTs, $X_1(k)$ (the DFT of even samples) and $X_2(k)$ (the DFT of odd samples). The only thing now to sort out is how these two DFTs are combined.

In order to do the combining, we need the 2-point butterfly operation which is written as:

$$\begin{bmatrix} X(0) \\ X(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

This operation is shown diagrammatically as:

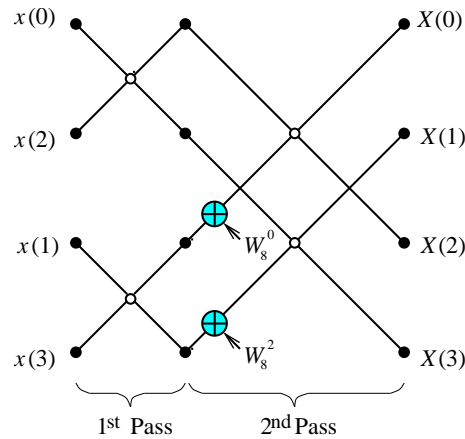


Now if we consider the FFT bin $X(k+2)$, we obtain:

$$X(k+2) = X_1(k+2) + W_4^{(k+2)} X_2(k+2) = X_1(k) - W_4^k X_2(k) \quad (2)$$

This result follows since $X_1(k)$ and $X_2(k)$ are 2-point DFTs with period 2 and $W_4^k = -W_4^{(k+2)}$. If we consider equations (1) and (2), we can see that the outputs $X(k)$ and $X(k+2)$ are formed by a 2-point butterfly operation on the inputs $X_1(k)$ and $W_4^k X_2(k)$. Therefore, we can form all four FFT outputs using two sets of 2-point butterfly operations for index values $k = 0$ and 1.

The flowchart of the 4-point FFT structure is shown below.



10.2

Radix 2 8-point FFT

The eight point DFT equation may be written as:

$$X(k) = \sum_{n=0}^7 x(n) W_8^{nk}$$

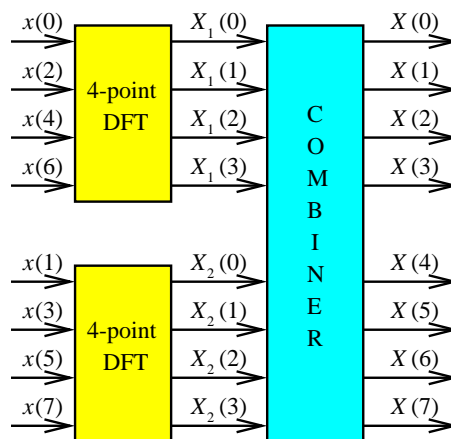
Following Example 10.1 of the textbook, we may write this as:

$$X(k) = \sum_{m=0}^3 x(2m)W_8^{2mk} + \sum_{m=0}^3 x(2m+1)W_8^{(2m+1)k}$$

This may again be simplified to give:

$$X(k) = \sum_{m=0}^3 x(2m)W_4^{mk} + W_8^k \sum_{m=0}^3 x(2m+1)W_4^{mk} = X_1(k) + W_8^k X_2(k) \tag{3}$$

We now recognise that $X(k)$ is the sum of two 4-point FFTs, $X_1(k)$ and $X_2(k)$, whose structure we have derived in answer to question 10.1. A diagram of the system we have obtained is shown below.



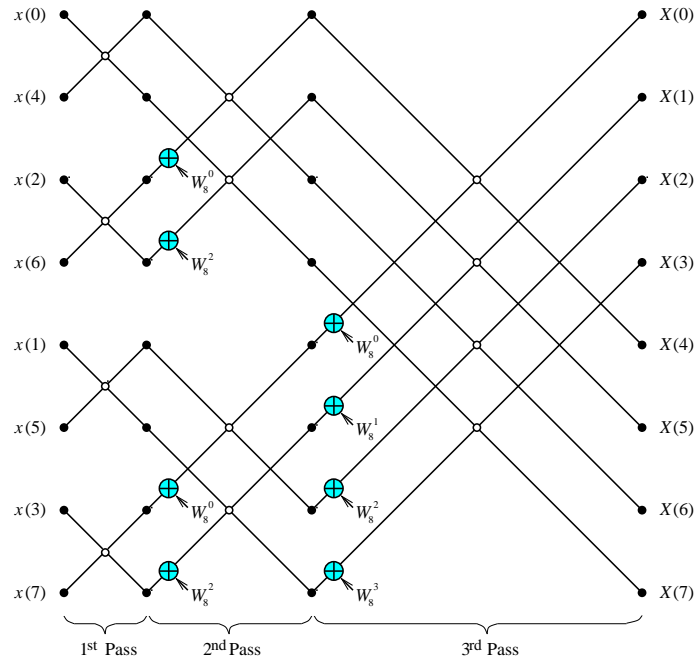
The only remaining thing to do is to derive the combiner that yields the full FFT. We should now consider the FFT output $X(k+4)$:

$$X(k+4) = X_1(k+4) + W_8^{(k+4)} X_2(k+4) = X_1(k) - W_8^k X_2(k) \tag{4}$$

Again, this result holds because X_1 and X_2 have period 4 (being 4-point FFTs). Similarly

$W_8^{(k+4)} = -W_8^k$. If we consider equations (3) and (4), we can see that the outputs $X(k)$ and $X(k + 4)$ are formed by a 2-point butterfly operation on the inputs $X_1(k)$ and $W_8^k X_2(k)$. Therefore, we can form all eight FFT outputs using four sets of 2-point butterfly operations for index values $k = 0, 1, 2$ and 3 .

The flowchart of the 8-point FFT is shown below. The first two passes are obtained from question 10.1, the third pass is derived above.



This question also asks about extending the combiner structure to a 16-point FFT. By analogy with the 8-point FFT derived above, the 16-point FFT can be split into two 8-point FFTs which process even ($X_e(k)$) and odd samples ($X_o(k)$) of $x(n)$. Each 8-point FFT may be implemented using the 8-point FFT structure we have derived above. The two sets of 8-point FFT outputs are then combined to form the 16-point FFT in a final combiner stage.

We can write two equations for the 16-point FFT $X(k)$ and $X(k + 8)$ in terms of the even 8-point FFT $X_e(k)$ and the odd 8-point FFT $X_o(k)$. These equations are:

$$X(k) = X_e(k) + W_{16}^k X_o(k)$$

$$X(k + 8) = X_e(k) - W_{16}^k X_o(k)$$

These two equations are derived in exactly the same way as equations (1), (2) for the 4-point FFT in 10.1 and (3), (4) for the 8-point FFT above. This the final combiner stage of the 16-point FFT consists of eight 2-point butterfly operations. The butterfly inputs are $X_e(k)$ and $W_{16}^k X_o(k)$ for indexes $k = 0 \dots 7$.

10.3

The intermediate outputs from the first pass DIT FFT in Figure 10.5 are given as:

$$x(0) + x(4)$$

$$x(0) - x(4)$$

$$x(2) + x(6)$$

$$\begin{aligned}
 &x(2) - x(6) \\
 &x(1) + x(5) \\
 &x(1) - x(5) \\
 &\text{etc.} \dots
 \end{aligned}$$

and the intermediate outputs (on the second pass) are then:

$$\begin{aligned}
 &x(0) + x(4) + W^0 [x(2) + x(6)] \\
 &x(0) - x(4) + W^2 [x(2) - x(6)] \\
 &x(0) + x(4) - W^0 [x(2) - x(6)] \\
 &x(0) - x(4) - W^2 [x(2) - x(6)] \\
 &x(1) + x(5) + W^0 [x(3) + x(7)] \\
 &x(1) - x(5) + W^2 [x(3) - x(7)] \\
 &x(1) + x(5) - W^0 [x(3) - x(7)] \\
 &x(1) - x(5) - W^2 [x(3) - x(7)]
 \end{aligned}$$

The third pass outputs then obtain the full 8-point transform output values.

10.4

For the radix-4 processor with inputs $x(0) = a + jb$, $x(1) = c + jd$, $x(2) = e + jf$, $x(3) = g + jh$ the DFT outputs are given by equation (10.16) as:

$$\begin{aligned}
 X(0) &= x(0) + x(1) + x(2) + x(3) \\
 &= [a + c + e + g] + j [b + d + f + h] \\
 X(1) &= x(0) - j x(1) - x(2) + j x(3) \\
 &= a + jb - j(c + jd) - (e + jf) + j(g + jh) \\
 &= [a + d - e - h] + j [b - c - f + g] \\
 X(2) &= x(0) - x(1) + x(2) - x(3) \\
 &= [a - c + e - g] + j [b - d + f - h] \\
 X(3) &= x(0) + j x(1) - x(2) - j x(3) \\
 &= [a - d - e + h] + j [b + c - f - g]
 \end{aligned}$$

Note that even transformed samples are obtained by only additions and subtractions of real and imaginary input sample values. The odd transformed values involve complex multiplication but, for the radix-4, this only involves interchanging real and imaginary (c/d and g/h) sample values followed by addition or subtraction operations!

10.5

The intermediate outputs from the first pass are given as:

$$\begin{aligned}
 &x(0) + x(4) \\
 &W^a [x(0) - x(4)] \\
 &x(2) + x(6)
 \end{aligned}$$

$$\begin{aligned}
 &W^b [x(2) - x(6)] \\
 &x(1) + x(5) \\
 &W^c [x(1) - x(5)] \\
 &\text{etc. . . .}
 \end{aligned}$$

Thus the intermediate outputs on the second pass are:

$$\begin{aligned}
 &x(0) + x(4) + x(2) + x(6) \\
 &W^a [x(0) - x(4)] + W^b [x(2) - x(6)] \\
 &W^e [x(0) + x(4) - x(2) - x(6)] \\
 &W^f W^a [x(0) - x(4)] - W^f W^b [x(2) - x(6)] \\
 &x(1) + x(5) + x(3) + x(7) \\
 &W^c [x(1) - x(5)] + W^d [x(3) - x(7)] \\
 &W^g [x(1) + x(5) - x(3) - x(7)] \\
 &W^h W^c [x(1) - x(5)] - W^h W^d [x(3) - x(7)]
 \end{aligned}$$

Now for example, for the full 8-point transform outputs comprise:

$$\begin{aligned}
 X(0) &= x(0) + x(4) + x(2) + x(6) + x(1) + x(5) + x(3) + x(7) \\
 X(1) &= W^a [x(0) - x(4)] + W^b [x(2) - x(6)] + W^c [x(1) - x(5)] + W^d [x(3) - x(7)] \\
 X(2) &= W^e [x(0) + x(4) - x(2) - x(6)] + W^g [x(1) + x(5) - x(3) - x(7)] \\
 X(3) &= W^f [W^a [x(0) - x(4)] - W^b [x(2) - x(6)]] + W^h [W^c [x(1) - x(5)] - W^d [x(3) - x(7)]] \\
 X(4) &= W^i [x(0) + x(4) + x(2) + x(6) - x(1) - x(5) - x(3) - x(7)] \\
 X(5) &= W^j [W^a [x(0) - x(4)] + W^b [x(2) - x(6)] - W^c [x(1) - x(5)] + W^d [x(3) - x(7)]] \\
 X(6) &= W^k [W^e [x(0) + x(4) - x(2) - x(6)] - W^g [x(1) + x(5) - x(3) - x(7)]] \\
 X(7) &= W^l [W^f [W^a [x(0) - x(4)] - W^b [x(2) - x(6)]] - W^h [W^c [x(1) - x(5)] + W^d [x(3) - x(7)]]]
 \end{aligned}$$

Now recall the DFT equation (9.17):

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \cdot \\ \cdot \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ 1 & W_N^1 & W_N^2 & \cdot & \cdot & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdot & \cdot & W_N^{N-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & W_N^{N-1} & W_N^{N-2} & \cdot & \cdot & W_N^1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \cdot \\ \cdot \\ x(N-1) \end{bmatrix}$$

And the DFT equation (10.10) with the input data reordered as in the figure supplied is.

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 & W^0 & W^0 & W^0 & W^0 & W^0 \\ W^0 & W^4 & W^2 & W^6 & W^1 & W^5 & W^3 & W^7 \\ W^0 & W^0 & W^4 & W^4 & W^2 & W^2 & W^6 & W^6 \\ W^0 & W^4 & W^6 & W^2 & W^3 & W^7 & W^1 & W^5 \\ W^0 & W^0 & W^0 & W^0 & W^4 & W^4 & W^4 & W^4 \\ W^0 & W^4 & W^2 & W^6 & W^5 & W^1 & W^7 & W^3 \\ W^0 & W^0 & W^4 & W^4 & W^6 & W^6 & W^2 & W^2 \\ W^0 & W^4 & W^6 & W^2 & W^7 & W^3 & W^5 & W^1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \\ x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix}$$

The FFT matrix of the lower equation (10.10) has the corresponding columns reordered compared to the DFT equation.

Clearly from $X(1)$ we see that $W^a = W^0 = 1$, and we see that $W^b = W^2 = -1$, $W^c = W^1$ and $W^d = W^3$.

Similarly from $X(2)$ we see that $W^e = W^0$ and $W^g = W^2$.

Also from $X(3)$ we see that $W^f W^a = W^f W^0 = W^0$ thus $W^f = W^0$ and similarly $W^h = W^2$.

Each of the final equations $X(4)$ to $X(7)$ gives the concluding results that W^i, W^j, \dots, W^l all equal 1.

