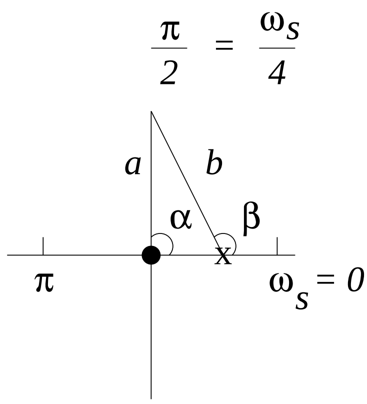
CHAPTER 5 - IIR Filters

5.1.

$$(i) \quad G(z) = \frac{z}{z - 0.5}$$

(a) $G(z)$ has a zero at $z = 0$ and a pole at $z = 0.5$.



(b) The only pole is inside the unit circle, therefore the filter is stable.

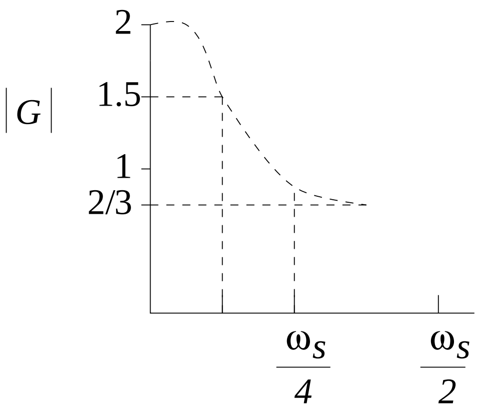
(c)

Magnitude at $\frac{\omega_s}{4}$

$a = 1$

$$b = \sqrt{1 + 0.5^2} = \sqrt{\frac{5}{4}}$$

$$\frac{a}{b} = 0.8944$$

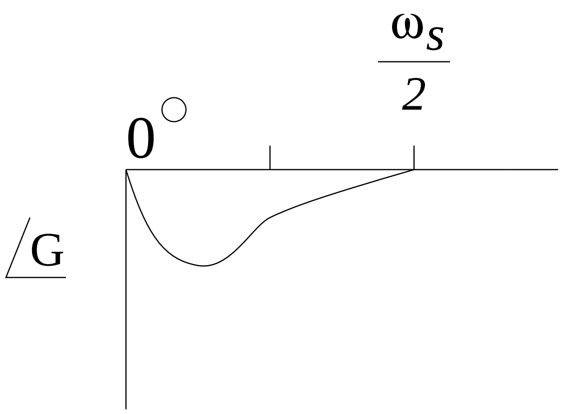


Phase at $\frac{\omega_s}{4}$

$\alpha = 90°$

$\beta = 180 - \tan^{-2} 2$

$\alpha - \beta = -26.57°$



(d)

$$G(z) = \frac{Y(z)}{X(z)} = \frac{z}{z - 0.5} = \frac{1}{1 - 0.5\, z^{-1}}$$

$$\Rightarrow \; Y(z)\;(1 - 0.5\, z^{-1}) = X(z)$$

$$\Rightarrow \; y(n\Delta t) = x(n\Delta t) + 0.5\, y(n\Delta t - \Delta t)$$

(e)

$x(0) = 1 \quad y(0) = 1$

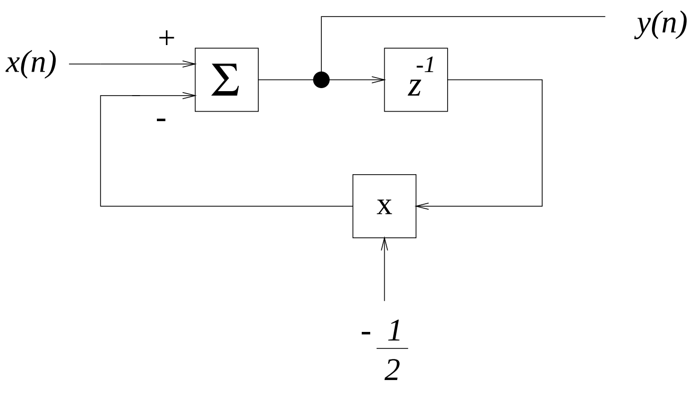
$x(\Delta t) = 0 \quad y(\Delta t) = 0.5$

$x(2\Delta t) = 0 \quad y(2\Delta t) = 0.25$

$x(3\Delta t) = 0 \quad y(3\Delta t) = 0.125$

$x(4\Delta t) = 0 \quad y(4\Delta t) = 0.0625$

(f)



5.2.

$$G(z) = \frac{z^2}{z^2 + ½}$$

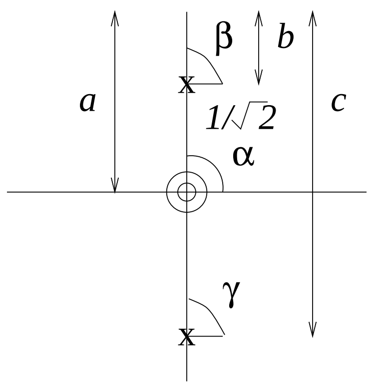
(a)

$$G(z) = \frac{z^2}{\left(z + \frac{j}{\sqrt{2}}\right)\left(z - \frac{j}{\sqrt{2}}\right)}$$

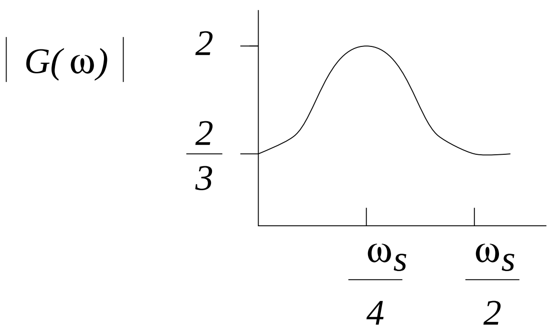
two zeros at $z = 0$ and poles at $z = \pm \frac{j}{\sqrt{2}}$.

(b) The poles are within the unit circle so the filter is stable.

(c)



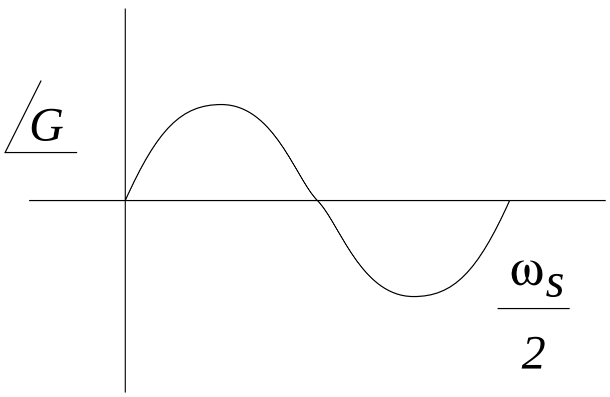
$|G(\omega)|$ at $\frac{\omega_s}{4}$



$a = 1$

$b = 1 - \frac{1}{\sqrt{2}}$

$c = 1 + \frac{1}{\sqrt{2}}$



$\frac{a^2}{bc} = 2$

phase is $\alpha + \alpha - \beta - \gamma = 90 + 90 - 90 - 90 = 0°$

(d)

$$G(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{z^2 + ½} = \frac{1}{1 + ½ \ z^{-2}}$$

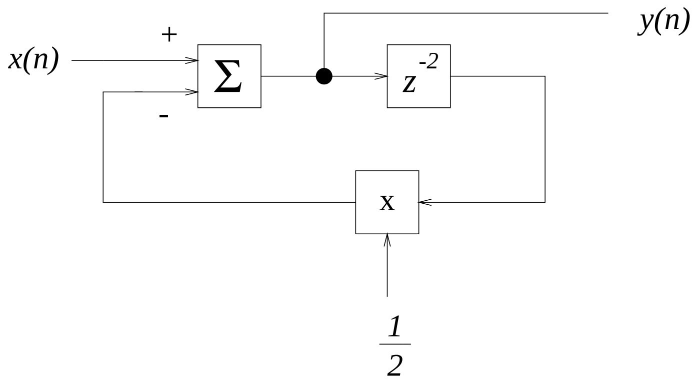
$$\Rightarrow \quad Y(z)\ (1 + 0.5\, z^{-2}) = X(z)$$

$$\Rightarrow \quad y(n\Delta t) = x(n\Delta t) - 0.5\, y(n\Delta t - 2\Delta t)$$

(e)

$x(0) = 1$	$y(0) = 1$
$x(\Delta t) = 0$	$y(\Delta t) = 0$
	$y(2\Delta t) = -0.5$
	$y(3\Delta t) = 0$
	$y(4\Delta t) = 0.25$

(f)



5.3.

The analogue prototype filter is given by:

$$H\ (s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

The prewarping substitution of equation (5.16) is first applied to the critical cut-off frequency $\omega_d = 10^2$ rad/s to yield:

$$\omega_a = \left\{ \frac{2}{6.28 \times 10^{-3}} \right\} \tan\,[10^2 \times \left\{ \frac{6.28\ .\ 10^{-3}}{2} \right\}] = 103.3 \text{ rad/s}$$

This figure is then used to denormallise the above prototype filter function giving:

$$H\ (s_d) = \frac{1}{\left\{ \dfrac{s}{103.3} \right\}^2 + \dfrac{\sqrt{2}s}{103.3} + 1}$$

The bilinear z-transform of equation (5.19) is now applied to the above equation to yield the equivalent digital filter transfer function:

$$H(z) = \frac{0.067z^{-2} + 0.135z^{-1} + 0.067}{0.413z^{-2} - 1.141\, z^{-1} + 1}$$

The difference equation is thus

$y(n) = 0.067x(n) + 0.135x(n-1) + 0.067x(n-2) + 1.141y(n-1) - 0.413y(n-2).$

The physical realisation of this design can be plotted in MATLAB^{TM} to obtain the actual magnitude and phase responses from this filter.

5.4.

$$G(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

$$\omega_s = 2\pi \times 8 \text{ kHz}$$

$$\omega_c = 2\pi \times 2 \text{ kHz} = \frac{\omega_s}{4} = \frac{\pi}{2\Delta t}$$

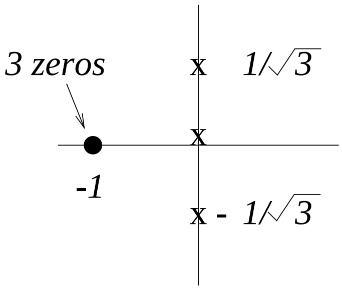
$$\omega_{csp} = \frac{2}{\Delta t} \tan\left(\omega_c \frac{\Delta t}{2} \right) = \frac{2}{\Delta t} \tan\left(\frac{\pi}{4} \right) = \frac{2}{\Delta t}$$

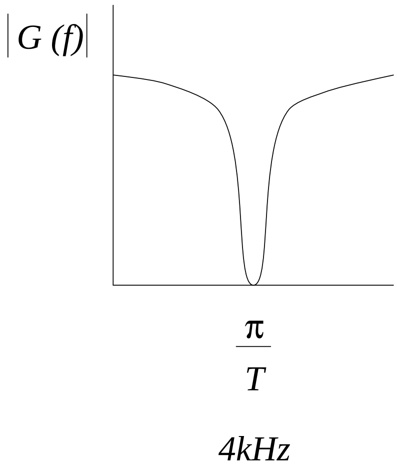
$$\Rightarrow G(s') = \frac{1}{\left(\frac{s'\Delta t}{2} \right)^3 + 2\left(\frac{s'\Delta t}{2} \right)^2 + s'\Delta t + 1}$$

$$\Rightarrow G(z) = \frac{1}{\left(\frac{z-1}{z+1} \right)^3 + 2\left(\frac{z-1}{z+1} \right)^2 + 2\left(\frac{z-1}{z+1} \right) + 1}$$

$$= \frac{(z+1)^3}{6z\left(z^2 + \frac{1}{3} \right)}$$

$$= \frac{(z+1)^3}{6z\left(z + j\frac{1}{\sqrt{3}} \right)\left(z - j\frac{1}{\sqrt{3}} \right)}$$

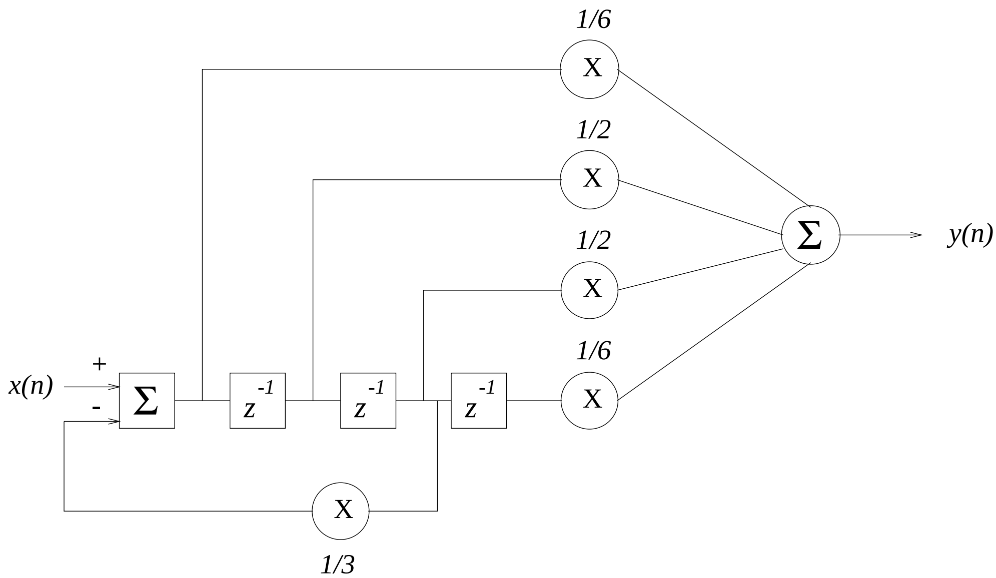




$$G(z) = \frac{(z+1)^3}{6z\left(z + j\frac{1}{\sqrt{3}}\right)\left(z - j\frac{1}{\sqrt{3}}\right)}$$

$$= \frac{z^3 + 3z^2 + 3z + 1}{6z^3 + 2z}$$

$$= \frac{\frac{1}{6} + \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{6}z^{-3}}{1 + \frac{1}{3}z^{-2}}$$



For a 3rd order filter the roll-off rate is 60 dB/decade.

5.5.(i)

$$\omega_{csp} = \frac{2}{\Delta t}\tan\left(\omega' \frac{\Delta t}{2}\right)$$

$\omega_{csp} \ = \ \omega'$ when ω' is small

$$e.g. \quad \omega' = \frac{\omega_s}{100} \Rightarrow \tau = \frac{100}{\omega_s}$$

$$F(z) = \frac{\frac{2}{\Delta t}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\tau}{1+\frac{2\tau}{\Delta t}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

using earlier analysis $\dfrac{2\tau}{\Delta t} = \dfrac{100}{\pi}$

$$F(z) = \frac{100\,(1-z^{-1})}{(\pi+100)+z^{-1}\,(\pi-100)}$$

5.5.(ii)

$$F(s) = \frac{s\tau}{1+s\,\tau}$$

$$\omega_{csp} = \frac{2}{\Delta t}\tan\left(\frac{\omega_s}{4}\times\frac{\Delta t}{2}\right)$$

$$= \frac{2}{\Delta t}\tan\left(\frac{\pi}{4}\right) = \frac{2}{\Delta t} = \frac{1}{\tau}$$

$$F(z) = \frac{\frac{2}{\Delta t}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\frac{\Delta t}{2}}{1+\frac{2}{\Delta t}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\frac{\Delta t}{2}}$$

$$= \frac{z-1}{2z}$$