CHAPTER 6 - FIR Filters

6.1

Passband edge at 1.25 kHz

For 40 dB rejection need Hanning window transition bandwidth = $\dfrac{3.1}{N\Delta t}$

where $\Delta t = 10^{-4}$ and $N = 21$

Therefore transition bandwidth = $\dfrac{3.1}{21} \times 10^4 = 1.48 \times 10^3 = 1.48$ kHz

Stopband starts at 1.25 + 1.48 = 2.73 kHz and goes up to 5 kHz (folding frequency), OR possibly 10–2.73 = 7.27 kHz.

6.2

$$1.25 \text{ kHz} = \frac{f_s}{8}$$

Now using $\omega = 2\pi f$ and $d\omega = 2\pi df$:-

$$c_n = \frac{2}{f_s} \int_0^{\frac{f_s}{2}} H_D(f) \cos \frac{2\pi nf}{f_s}\, df$$

$$= \frac{2}{f_s} \int_0^{\frac{f_s}{2}} 4 \cos \frac{2\pi nf}{f_s}\, df$$

$$= \left[4 \frac{2}{f_s} \frac{f_s}{2\pi n} \sin \frac{2\pi nf}{f_s} \right]_0^{\frac{f_s}{8}}$$

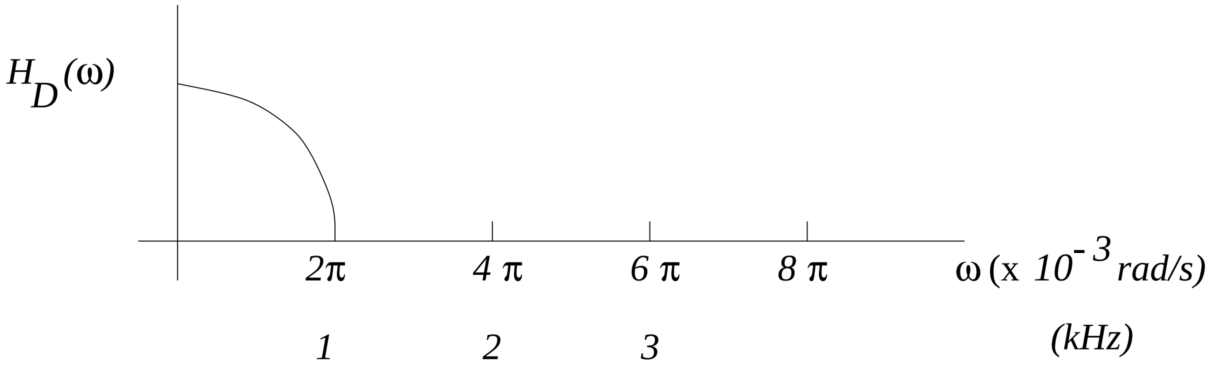
$$= \frac{4 \sin \frac{2\pi n f_s}{8 f_s}}{\pi n} = \frac{\sin \frac{\pi n}{4}}{\frac{\pi n}{4}}$$

$$\begin{array}{llll}
c_0 & = 1 & = a_{10} & \\
c_1 & = 0.9 & = a_9 & = a_{11} \\
c_2 & = 0.64 & = a_8 & = a_{12} \\
c_3 & = 0.3 & = a_7 & = a_{13} \\
c_4 & = 0 & = a_6 & = a_{14} \\
c_5 & = -0.18 & = a_5 & = a_{15} \\
c_6 & = -0.212 & = a_4 & = a_{16} \\
c_7 & = -0.128 & = a_3 & = a_{17} \\
c_8 & = 0 & = a_2 & = a_{18}
\end{array}$$

$$\begin{array}{llll} c_9 & = 0.1 & = a_1 & = a_{19} \\ c_{10} & = 0.128 & = a_0 & = a_{20} \end{array}$$

These values would not be used in practice without including a window function!

6.3.



Time between samples

$$\Delta t = \frac{1}{8 \times 10^3} \text{ s}$$

Sampling frequency $\omega_s = 2\pi \, (8 \times 10^3)$ rad/s

$$H_D(\omega) = \begin{cases} \cos(2\,\omega\,\Delta t), & \omega < \dfrac{\pi}{4\Delta t} \\ 0, & \text{elsewhere} \end{cases}$$

Impulse response calculation

$$c_n = \frac{\Delta t}{\pi} \int_0^{\frac{\pi}{\Delta t}} H_D(\omega) \cos(n\,\omega\,\Delta t) \; d\omega$$

$$= \frac{\Delta t}{\pi} \int_0^{\frac{\pi}{4\Delta t}} \cos(2\,\omega\,\Delta t) \cos(n\,\omega\,\Delta t) \; d\omega$$

$$= \frac{\Delta t}{2\pi} \int_0^{\frac{\pi}{4\Delta t}} \{ \cos((2+n)\,\omega\Delta t) + \cos((2-n)\,\omega\Delta t) \} \; d\omega$$

$$= \frac{\Delta t}{2\pi} \left[\frac{\sin((2+n)\,\omega\Delta t)}{(2+n)\,\Delta t} + \frac{\sin((2-n)\,\omega\Delta t)}{(2-n)\,\Delta t} \right]_0^{\frac{\pi}{4\Delta t}}$$

$$= \frac{1}{2\pi} \left[\frac{\sin\left((2+n)\,\frac{\pi}{4}\right)}{(2+n)} + \frac{\sin\left((2-n)\,\frac{\pi}{4}\right)}{(2-n)} \right]$$

$$= \frac{1}{8} \left[\text{sinc}\left((2+n)\,\frac{\pi}{4}\right) + \text{sinc}\left((2-n)\,\frac{\pi}{4}\right) \right]$$

Therefore $c_0 = 0.16$

$c_1 = 0.15$

$c_2 = 0.125$ (sinc 0 = 1 !)

$c_3 = 0.09$ *etc*

Apply Hanning window

Filter has $2M + 1 = 15$ taps

=> $M = 7$

$$w_n = \frac{1}{2} \left\{ 1 + \cos\left(\frac{n\pi}{7}\right) \right\}$$

$$c'_n = w_n \; c_n$$

Therefore

$c'_0 = 1 \times 0.16 = 0.16$

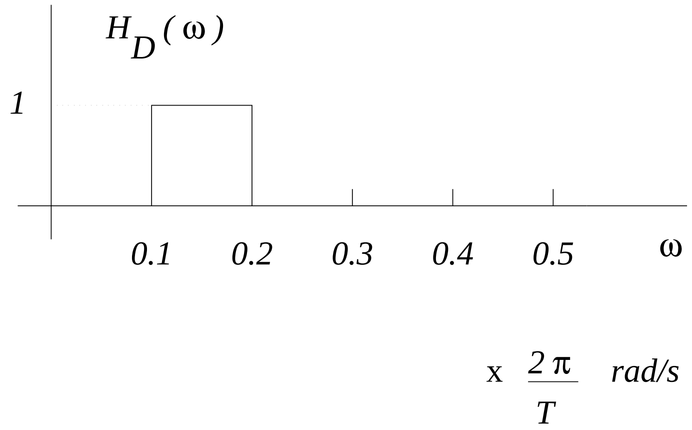
$c'_1 = 0.95 \times 0.15 = 0.14$

$c'_2 = 0.81 \times 0.125 = 0.10$

$c'_3 = 0.61 \times 0.09 = 0.05$

etc.

6.4.



$$c_n = \frac{\Delta t}{\pi} \int_{\frac{\pi}{5\Delta t}}^{\frac{2\pi}{5\Delta t}} \cos\ (n\ \omega\ \Delta t)\ \ d\omega$$

$$= \frac{\Delta t}{\pi} \left[\frac{\sin\ (n\ \omega\ \Delta t)}{n\ \Delta t} \right]_{\frac{\pi}{5\Delta t}}^{\frac{2\pi}{5\Delta t}}$$

$$= \frac{1}{\pi} \left\{ \frac{\sin\ (\frac{2\pi n}{5})}{n} - \frac{\sin\ (\frac{\pi n}{5})}{n} \right\}$$

Apply Hanning window

The filter has $(2M + 1) = 17$ taps

=> $M = 8$

$$w_n = \frac{1}{2} \left\{ 1 + \cos\ \frac{(n\pi)}{8} \right\}$$

$$c'_n = w_n\ c_n$$

$$c'_0 = 1.0 \times 0.2 = 0.2$$

$$c'_1 = 0.96 \times 0.116 = 0.112$$

$$c'_2 = 0.85 \times -0.058 = -0.05$$

etc.

6.5.

For a low-pass filter (0 – $f_s/5$)

$$c_n = \frac{\Delta t}{2\pi} \int_0^{2\pi/\Delta t} H_D(\omega) \cos \frac{2\pi n\omega}{\omega_s} \, d\omega$$

$$= \frac{\Delta t}{\pi} \int_0^{\omega_s/5} \sqrt{10} \cos \frac{2\pi n\omega}{\omega_s} \, d\omega$$

$$= \left[\frac{\Delta t}{\pi} \sqrt{10} \frac{\omega_s}{2\pi n} \sin \frac{2\pi n\omega}{\omega_s} \right]_0^{\omega_s/s}$$

$$= \sqrt{10} \frac{\sin \dfrac{2\pi n\omega_s}{5\,\omega_s}}{\pi\, n} = \frac{2\sqrt{10}}{5} \frac{\sin \dfrac{2\pi n}{5}}{\dfrac{2\pi n}{5}}$$

$$= 1.265 \frac{\sin \dfrac{2\pi n}{5}}{\dfrac{2\pi n}{5}} \text{ or } \left(1.006 \frac{\sin \dfrac{2\pi n}{5}}{n} \right)$$

For unweighted filters substitution for n yields:

n	=	0,	1,	2,	3,	4,	5,	6
c_{n1}	=	1.265,	0.95,	0.30,	–0.20,	–0.24,	0,	0.16

For narrowband low-pass filter (0 – $f_s/10$)

$$c_n = \left[\frac{\Delta t}{2\pi} \sqrt{10} \frac{\omega_s}{2\pi n} \sin \frac{2\pi n\omega}{\omega_s} \right]_0^{\omega_s/10}$$

$$= \sqrt{10} \sin \frac{\dfrac{2\pi n\omega_s}{10\,\omega_s}}{\pi\, n} = \frac{\sqrt{10}}{5} \frac{\sin \dfrac{2\pi n}{10}}{\dfrac{2\pi n}{10}}$$

and again substitution for an unweighted filter gives:

n	=	0,	1,	2,	3,	4,	5,	6
c_{n2}	=	0.632,	0.59,	0.48,	0.32,	0.15,	0,	–0.103

The number of taps required for $H_2(\omega)$ is determined by the transition bandwidth of this filter. If the transition band is the same as for $H_1(\omega)$, 31 taps would suffice for $H_2(\omega)$ as well. Alternatively, if the transition bandwidth for $H_2(\omega)$ was one half of that for $H_1(\omega)$, the $H_2(\omega)$ filter will need twice the number of coefficients, i.e. it would be a 63 tap design.

Combining these filters gives:

n	=	0,	1,	2,	3,	4,	5,	6, etc
c_{n1}	=	1.265,	0.95,	0.296,	–0. 197,	–0. 239,	0,	0.16
$-c_{n2}$	=	0.632,	0.59,	0.48,	0.32,	0.15,	0,	–0. 10
$c_{n1} - c_{n2}$	=	0.632,	0.36	–0. 18,	–0. 517,	–0. 389,	0,	0.26

Now these weights now have to be converted into a symmetrical filter, where $n = 0$ gives the centre tap weight value.