CHAPTER 9 - DFT

9.1.

$$X\ (\omega)\ =\ \sum_{n=-\infty}^{\infty}\ x(n)\ e^{-j\omega n\Delta t}\ =\ \frac{1}{4}+\frac{1}{2}\ e^{-j\omega\Delta t}\ +\ \frac{1}{4}\ e^{-j2\omega\Delta t}$$

$$=\left\{\frac{1}{2}+\frac{1}{4}\ e^{j\omega\Delta t}\ +\ \frac{1}{4}\ e^{-j\omega\Delta t}\right\}\ e^{-j\omega\Delta t}\ =\ \frac{1}{2}\ \{\ 1+\cos\ \ \omega\Delta t\ \}\ e^{-j\omega\Delta t}$$

Replacing $\omega\Delta t$ with $\frac{2\pi k}{3}$

$$X(k)\ =\ \frac{1}{2}\left\{\ 1+\cos\left(\frac{2\pi k}{3}\right)\right\}e^{-j\frac{2\pi k}{3}}$$

$$=>\ \ X(0)\ =\ 1\ ;\ X(1)=\frac{1}{4}\ e^{-j\frac{2\pi}{3}}\ \ ;\ \ X(2)=\frac{1}{4}\ e^{-j\frac{4\pi}{3}}$$

9.2

$$X(\omega)\ =\ \sum_{n=0}^{N}\ x(n)\ e^{-jn\omega\Delta t}\ =\ \sum_{n=0}^{L-1}\ e^{-jn\omega\Delta t}\ =\ \frac{1-e^{-j\omega\Delta tL}}{1-e^{-j\omega\Delta t}}$$

(Geometric series ....)

$$=\ \frac{e^{-j\omega\Delta tL/2}}{e^{-j\omega\Delta t/2}}\ \frac{\left\{e^{j\omega t\Delta L/2}-e^{-j\omega t\Delta L/2}\right\}}{\left\{e^{j\omega\Delta t/2}-e^{-j\omega\Delta t/2}\right\}}$$

$$=\ e^{-j\omega\Delta t(L-1)/2}\ \frac{\sin\left(\frac{\omega\Delta tL}{2}\right)}{\sin\left(\frac{\omega\Delta t}{2}\right)}$$

Either making the substitution $\frac{2\pi k}{N}=\omega\Delta t$, or using

$$X(k)\ =\ \sum_{n=0}^{L-1}\ e^{-j\frac{2\pi nk}{N}}$$

we obtain,

$$X(k)\ =\ e^{-j\ \frac{\pi k(L-1)}{N}}\ \frac{\sin\left(\frac{\pi kL}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)}$$

9.3

Frequency resolution of DFT is $\frac{f_s}{N} = \frac{10\times 10^3}{200}$ = 50 Hz.

But difference Δf = 40 Hz so they cannot be resolved.
For resolution Δf must = 50 Hz and both frequencies must be a multiple of 50 Hz, i.e. 1600 Hz and 1650 Hz tones are required for example to resolve them as separate DFT output signals!

9.4.

For an 8-point DFT the coefficient matrix is:-

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ 1 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ 1 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\ 1 & . & . & . & . & . & . & . \\ 1 & W_8^7 & W_8^6 & W_8^5 & W_8^4 & W_8^3 & W_8^2 & W_8^1 \end{bmatrix}$$

Fourier coefficients are obtained from

$$W_N^{nk} = e^{-j\,\frac{n\pi\ nk}{N}} \quad \text{where} \quad N = 8$$

k = component no. (0 − 7)

n = weight no.

For $k = 0$, $e^{-j0} = 1$ irrespective of n value.
For $k = 1$, coefficients are:

$$e^{\frac{-j\,0}{8}},\ e^{-j\,\frac{2\pi}{8}},\ e^{-j\frac{2\pi\ 2}{8}},\ e^{-j\,\frac{2\pi\ 3}{8}},\ e^{-j\,\frac{2\pi\ 4}{8}},\ e^{-j\,\frac{2\pi\ 5}{8}},\quad etc.$$

$$= 1,\ e^{-j\,\frac{\pi}{4}},\ e^{-j\,\frac{\pi}{2}},\ e^{-j\,\frac{3\pi}{4}},\ e^{-j\pi},\ e^{-j\,\frac{5\pi}{4}},\quad etc.$$

$$= 1,\ .7 - j\,.7,\ -j,\ -.7 - j\,.7,\ -1,\ +.7 + j\,.7,\ j,\quad etc.$$

The full matrix is then:-

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & .7 - j.7 & -j & -.7 - j.7 & -1 & -.7 + j.7 & j & .7 + j.7 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -.7 - j.7 & j & .7 - j.7 & -1 & .7 + j.7 & -j & -.7 + j.7 \\ 1 & . & . & . & . & . & . & . \\ 1 & .7 + j.7 & j & -.7 + j.7 & -1 & -.7 - j.7 & -j & .7 - j.7 \end{bmatrix}$$

The samples of $\cos\omega t$ and $\sin\omega t$ are:-

	$x(0)$	$x(1)$	$x(2)$	$x(3)$	$x(4)$	$x(5)$	$x(6)$	$x(7)$
cos	1.0	.707	0	−.707	−1.0	−.707	0	.707

sin   0   .707 1.0   .707   0   −.707 −1   −.707

The Fourier transformed output values of the above input data samples are then given as:-

$$X(0) = (1 + j0)\,[\,(1 + j0) + (.707 + j\,.707) + (\;\;) + \cdots + (\;\;)\text{ etc}]$$

$$= 0 + j0$$

$$X(1) = \sum_{0}^{7} (a + jb)\,(a - jb) = 8\,(a^2 + b^2) = 8$$

Note in this **special case** (zero starting phase and one cycle per block length) the sampled phasor in the 2nd row of the 8×8 DFT matrix is the complex conjugate of the input sampled phasor. This simplifies the calculation of $X(1)$ to the expression above.

$$X(2) = 0 + j\,0$$

etc.

9.5.

Now signal samples are given in question and we apply the DFT matrix again from problem 9.4.

For $X(0)$ we simply sum the 8 sample values.

$j4$, - 2 $\sqrt{2}$ + j 2 $\sqrt{2}$, - 4, - 2 $\sqrt{2}$ - j 2 $\sqrt{2}$, etc.

to get the answer = 0

as here all the DFT coefficients are +1.

For $X(1)$ we now multiply the signal values with the second row of the DFT matrix:

SIGNAL VALUES		SECOND DFT ROW		
$j4$	$\times$	1	$=$	$j4$
$-2\sqrt{2} + j2\sqrt{2}$	$\times$	$.7 - j7$	$=$	$-2 + 2 + 2j + 2j$
-4	$\times$	$-j$	$=$	$j4$
$-2\sqrt{2} - j2\sqrt{2}$	$\times$	$-.7 - j.7$	$=$	$+2 - 2 + 2j + 2j$
$-j4$	$\times$	-1	$=$	$j4$
$2\sqrt{2} - j2\sqrt{2}$	$\times$	$-.7 + j.7$	$=$	$-2 + 2 + 2j + 2j$
4	$\times$	j	$=$	$j4$
$2\sqrt{2} + 2\sqrt{2}$	$\times$	$.7 + j.7$	$=$	$+2 - 2 + 2j + 2j$

Total for $X(1) = 8 \times 4j = 32j$

For $X(2)$ we now multiply the signal values with the third row of the DFT matrix:

SIGNAL VALUES		THIRD DFT ROW		
$j4$	$\times$	1	$=$	$j4$
$-2\sqrt{2} + j2\sqrt{2}$	$\times$	$-j$	$=$	$+2\sqrt{2} + j2\sqrt{2}$
-4	$\times$	-1	$=$	4
$-2\sqrt{2} - j2\sqrt{2}$	$\times$	$+j$	$=$	$+2\sqrt{2} - j2\sqrt{2}$
$-j4$	$\times$	1	$=$	$-j4$
$+2\sqrt{2} - j2\sqrt{2}$	$\times$	$-j$	$=$	$-2\sqrt{2} - j2\sqrt{2}$
4	$\times$	-1	$=$	-4
$+2\sqrt{2} + j2\sqrt{2}$	$\times$	$+j$	$=$	$-2\sqrt{2} + j2\sqrt{2}$
				TOTAL SUMS TO ZERO for $X(2)$

Similarly for 3rd component sum = 0 and $X(3) = 0$.

Full DFT output is thus 0, $32j$, 0, 0, etc. As the input signal is complex the X(7) value will be zero and *not* $32j$.

9.6.

Consider Figure 9.19(b) as a Wiener filter problem to identify the coefficients $a_0, \cdots, a_n$. To do this we calculate the autocorrelation at the input to the filter and the cross correlation between the filter input and the desired outputs.

The Wiener equation thus becomes:

$$\begin{bmatrix} 1.03 & 0.31 \\ 0.31 & 1.03 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0.31 \\ -0.81 \end{bmatrix}$$

$$1.03\, a_1 + 0.31\, a_2 = 0.31$$

$$0.31\, a_1 + 1.03\, a_2 = -0.81$$

solution $a_1 = 0.5912$

$$a_2 = -0.9643$$

minimum MSE

$$\sigma_e^2 = E\left[\, x^2 \,\right] - \Phi_{xy}^T\, \mathbf{a}$$

$$= 1.03 - [\, 0.31 \; -0.81 \,] \begin{bmatrix} 0.5912 \\ -0.9643 \end{bmatrix}$$

$$= 0.0656$$

AR spectral estimate

$$\hat{S}_{xx}(\omega) = \frac{\sigma_e^2}{|\ 1 - a_1\ e^{-j\omega\Delta t} - a_2\ e^{-2j\omega\Delta t}\ |^2}$$

sampling frequency $\omega_s = \dfrac{2\pi}{\Delta t}$ rad/s

$$=> \quad \frac{\omega_s}{2} = \frac{\pi}{\Delta t} \text{ rad/s}$$

estimates

at $\omega = 0,\ \dfrac{\pi}{5\Delta t},\ \dfrac{2\pi}{5\Delta t},\ \dfrac{3\pi}{5\Delta t},\ \dfrac{4\pi}{5\Delta t},\ \dfrac{\pi}{\Delta t}$ rad/s

$$\omega = 0\ ,\ \hat{S}_{xx}(0) = \frac{0.0656}{|\ 1 - a_1 - a_2\ |^2}$$

$$= 0.0348$$

$$= -14.58 \text{ dB}$$

$$\omega = \frac{\pi}{5\Delta t}\ ,\ \hat{S}_{xx}\left(\frac{\pi}{5\Delta t}\right) = \frac{0.0656}{|\ 1 - a_1\ e^{-j\pi/5} - a_2\ e^{-2j\pi/5}\ |^2}$$

real part of denominator:

$$= 1 - a_1 \cos\left(\frac{\pi}{5}\right) - a_2 \cos\left(\frac{2\pi}{5}\right)$$

$$= 0.8197$$

imaginary part of denominator

$$= a_1 \sin\left(\frac{\pi}{5}\right) + a_2 \sin\left(\frac{2\pi}{5}\right)$$

$$= -0.5695$$

Therefore $\hat{S}_{xx}\left(\dfrac{\pi}{5}\right) = \dfrac{0.0656}{(0.8197)^2 + (0.5695)^2}$

$$= 0.0658$$

$$= -11.8 \text{ dB}$$

similarly

$$\hat{S}_{ss}\left(\frac{2\pi}{5\Delta t}\right) = 16.71 \text{ dB}$$

$$\hat{S}_{xx}\left(\frac{3\pi}{5\Delta t}\right) = -13.40 \text{ dB}$$

$$\hat{S}_{ss}\left(\frac{4\pi}{5\Delta t}\right) = -18.60 \text{ dB}$$

$$\hat{S}_{xx}\left(\frac{\pi}{\Delta t}\right) = -19.98 \text{ dB}$$

9.7.

$$\mathbf{R}_{xx} = \sum_n \begin{bmatrix} x(n) \\ x(n-1) \\ x(n-2) \end{bmatrix} [\, x(n)\ x(n-1)\ x(n-2)\,]$$

The easiest one to calculate is the co-variance form since it involves the smallest number of terms.

$$\mathbf{R}_{xx}\ (covar) = \begin{bmatrix} -0.5 \\ 2.0 \\ 3.0 \end{bmatrix} [\, -0.5\ \ 2.0\ \ 3.0\,]$$

$$+ \begin{bmatrix} 1.5 \\ -0.5 \\ 2.0 \end{bmatrix} [\, 1.5\ \ -0.5\ \ 2.0\,]$$

$$+ \begin{bmatrix} -1.0 \\ 1.5 \\ -0.5 \end{bmatrix} [\, -1.0\ \ 1.5\ \ -0.5\,]$$

i.e. it comprises the addition of 3 (3×3) matrices.

The pre-windowed form:

$$\mathbf{R}_{xx}\ (pre) = \mathbf{R}_{xx}\ (covar)$$

$$+ \begin{bmatrix} 3.0 \\ 0 \\ 0 \end{bmatrix} [\, 3.0\ \ 0\ \ 0\,]$$

$$+ \begin{bmatrix} 2.0 \\ 3.0 \\ 0 \end{bmatrix} [\, 2.0\ \ 3.0\ \ 0\,]$$

i.e. it requires the addition of 5 (3×3) matrices.

The post-windowed form:

$$\mathbf{R}_{xx}\ (post) = \mathbf{R}_{xx}\ (covar)$$

$$+ \begin{bmatrix} 0 \\ -1.0 \\ 1.5 \end{bmatrix} [\, 0 \quad -1.0 \quad 1.5 \,]$$

$$+ \begin{bmatrix} 0 \\ 0 \\ -1.0 \end{bmatrix} [\, 0 \quad 0 \quad -1.0 \,]$$

i.e. again this involves the addition of 5 (3×3) matrices.

The auto-correlation form:

$$\mathbf{R}_{xx}\ (auto) = \mathbf{R}_{xx}\ (pre)$$

$$+ \begin{bmatrix} 0 \\ -1.0 \\ 1.5 \end{bmatrix} [\, 0 \quad -1.0 \quad 1.5 \,]$$

$$+ \begin{bmatrix} 0 \\ 0 \\ -1.0 \end{bmatrix} [\, 0 \quad 0 \quad -1.0 \,]$$

i.e. now we need the addition of 7 (3×3) matrices.

9.8.

In using the covariance form, we do not want to make any assumptions about the signal before or after the samples we have available. At the same time we want to use as much of the data as possible in order to make the estimate of the prediction coefficient as good as possible.

Least squares solution (8.13) applied to linear prediction (9.32).

$$\mathbf{R}_{yy}\ \mathbf{a} = \mathbf{r}_{yx}$$

$$\mathbf{R}_{yy} = \sum_n \begin{bmatrix} x(n-1) \\ x(n-2) \end{bmatrix} [\, x(n-1)\ x(n-2) \,]$$

$$\mathbf{r}_{yx} = \sum_n \begin{bmatrix} x(n-1) \\ x(n-2) \end{bmatrix} x(n)$$

To form one term in $\mathbf{r}_{yx}$ we need 3 data samples: $-x(n)$, $x(n-1)$, $x(n-2)$.

Data available

$n =$ 0 1 2 3 4

$x(n) =$ 3.0 2.0 −0.5 1.5 −1.0

If we start the summations at $n = 2$ and finish at $n = 4$, we use as much data as possible and make no assumptions about the signal before or after the data we have available.
Thus

$$\mathbf{R}_{yy} = \sum_{n=2}^{4} \begin{bmatrix} x(n-1) \\ x(n-2) \end{bmatrix} [\ x(n-1)\ x(n-2)\ ]$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} [2\ 3] + \begin{bmatrix} -0.5 \\ 2 \end{bmatrix} [-0.5\ 2] + \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix} [1.5 - 0.5]$$

$$= \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} + \begin{bmatrix} 0.25 & -1 \\ -1 & 4 \end{bmatrix} + \begin{bmatrix} 2.25 & -0.75 \\ -0.75 & 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} 6.5 & 4.25 \\ 4.25 & 13.25 \end{bmatrix}$$

and

$$\mathbf{r}_{yx} = \sum_{n=2}^{4} \begin{bmatrix} x(n-1) \\ x(n-2) \end{bmatrix} x(n)$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} (-0.5) + \begin{bmatrix} -0.5 \\ 2 \end{bmatrix} (1.5) + \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix} (-1.0)$$

$$= \begin{bmatrix} -1 \\ -1.5 \end{bmatrix} + \begin{bmatrix} -0.75 \\ 3 \end{bmatrix} + \begin{bmatrix} -1.5 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} -3.25 \\ 2.0 \end{bmatrix}$$

To find coefficients we use;

$$\begin{bmatrix} 6.5 & 4.25 \\ 4.25 & 13.25 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -3.25 \\ 2.0 \end{bmatrix}$$

Solve simultaneous equations yields

$$a_1 = -0.758$$

$$a_2 = 0.394$$

9.9.

input data

$n =$	1	2	3	4	5
$y(n)$	1	1	-1	-1	1

Covariance from estimate

$$\mathbf{R}_{yy} = \sum_{n=2}^{5} \mathbf{y}(n)\, \mathbf{y}^T(n)$$

$$= \mathbf{y}(2)\, \mathbf{y}^T(2) + \mathbf{y}(3)\, \mathbf{y}^T(3) + \mathbf{y}(4)\, \mathbf{y}^T(4) + \mathbf{y}(5)\, \mathbf{y}^T(5)$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1\ 1] + \begin{bmatrix} -1 \\ 1 \end{bmatrix} [-1\ 1] + \begin{bmatrix} -1 \\ -1 \end{bmatrix} [-1\ -1] + \begin{bmatrix} 1 \\ -1 \end{bmatrix} [1\ -1]$$

$$= \begin{bmatrix} 1\ 1 \\ 1\ 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1\ 1 \\ 1\ 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4\ 0 \\ 0\ 4 \end{bmatrix}$$

output data

$n =$	1	2	3	4	5
$x(n)$	0.898	1.346	-0.415	-1.236	0.467

$$\mathbf{r}_{yx} = \sum_{n=2}^{5} \mathbf{y}(n)\, x(n)$$

$$= \mathbf{y}(2)\, x(2) + \mathbf{y}(3)\, x(3) + \mathbf{y}(4)\, x(4) + \mathbf{y}(5) x(5)$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} 1.346 + \begin{bmatrix} -1 \\ 1 \end{bmatrix} (-0.415) + \begin{bmatrix} -1 \\ -1 \end{bmatrix} (-1.236) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} 0.467$$

$$= \begin{bmatrix} 1.346 \\ 1.346 \end{bmatrix} + \begin{bmatrix} 0.415 \\ -0.415 \end{bmatrix} + \begin{bmatrix} 1.236 \\ 1.236 \end{bmatrix} + \begin{bmatrix} 0.467 \\ -0.467 \end{bmatrix}$$

$$= \begin{bmatrix} 3.464 \\ 1.7 \end{bmatrix}$$

Least squares solution is achieved by solving:

$$\mathbf{R}_{yy}\ \mathbf{h} = \mathbf{r}_{yx}$$

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} 3.464 \\ 1.7 \end{bmatrix}$$

$$h_0 = \frac{3.464}{4} = 0.866$$

$$h_1 = \frac{1.7}{4} = 0.425$$