Sinusoidal EM Waves

22.2MB1

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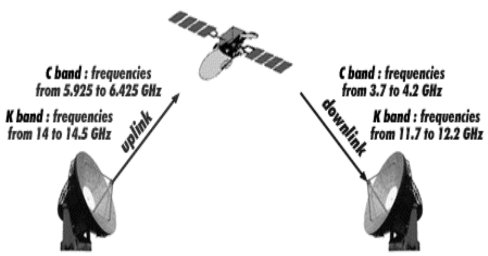
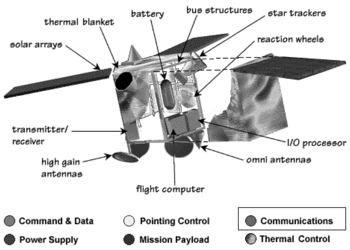
Introduction



Introduction



Introduction

How to design the communications side of a satellite?

1 Frequency ?
- Optimise propagation ? — Wave equations
- Optimise data rate ? — Signal analysis

2 Antennas?
- Directivity ? — Wave equations
- Gain ? — & Fourier analysis

3 Power budget?
- Linked to 1 & 2
- Poynting vector

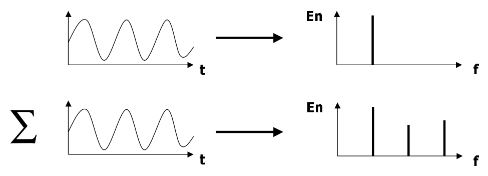
Section Contents

- Fourier representation
- Phasors
- Maxwell equations in phasor forms
- Plane wave propagation, phase velocity
- Power flow and Poynting vector
- Propagation in media
  - Propagation in conductive media
  - Propagation in Good dielectric
  - Propagation in Good conductor
  - Skin Depth
  - Power loss in metal

} Frequency analysis

Sinusoidal time variation

Time domain	Frequency domain
• Periodic signal: sum of harmonically related sinusoids.	• Energy at frequencies multiple of the fundamental frequency



Fourier series
Synthesis equations

For complex periodic signals

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_o t}$$

Harmonically related complex exponentials
Fourier Analysis

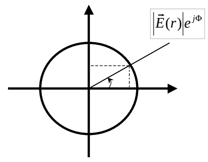
Sinusoidal EM Waves

- We are interested in the behaviour of sinusoidal waves of the form:

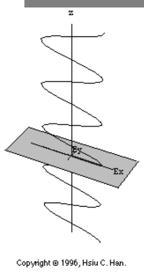
$$\vec{E}(r,t) = \mathrm{Re}\left(\vec{E}(r)e^{jwt}\right)$$

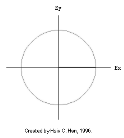
$\vec{E}(r)$ Phasor. Function of space only.

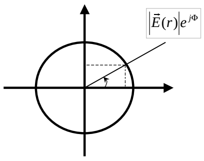
$$\vec{E}(r) = \left|\vec{E}(r)\right| e^{j\Phi}\vec{u}$$



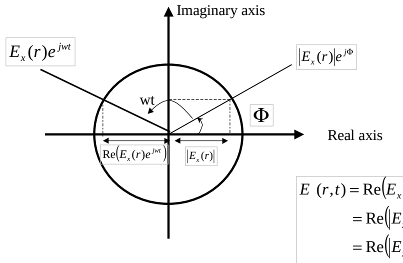
Sinusoidal EM Waves





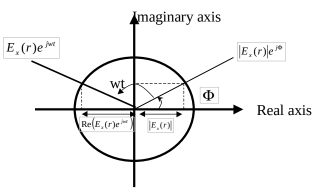


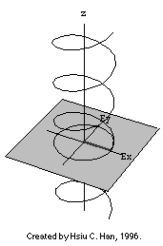
Geometric interpretation



$$\begin{aligned} E\ (r,t) &= \mathrm{Re}\left(E_x(r)e^{jwt}\right) \\ &= \mathrm{Re}\left(\left|E_x(r)\right|e^{j\Phi}e^{jwt}\right) \\ &= \mathrm{Re}\left(\left|E_x(r)\right|e^{j(wt+\Phi)}\right) \\ &= \left|E_x(r)\right|\cos(wt+\Phi) \end{aligned}$$

Geometric interpretation





Complex Numbers
Euler's relation

Euler's relation

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

$$\cos\omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad \sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

Maxwell's equations in phasor notation

Example

$$\vec{\nabla}\times\vec{H}(r,t)=\vec{J}(r,t)+\frac{\partial\vec{D}(r,t)}{\partial t}$$

Becomes

$$\vec{\nabla}\times\mathrm{Re}\left(\vec{H}(r)e^{jwt}\right)=\mathrm{Re}\left(\vec{J}(r)e^{jwt}\right)+\frac{\partial}{\partial t}\mathrm{Re}\left(\vec{D}(r)e^{jwt}\right)$$

$$\mathrm{Re}\left(\left\{\vec{\nabla}\times\vec{H}(r)-\vec{J}(r)-jw\vec{D}(r)\right\}e^{jwt}\right)=0$$

Finally we have

$$\vec{\nabla}\times\vec{H}=\vec{J}+jw\vec{D}$$

Maxwell's equations in phasor notation

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$
$$\vec{\nabla}.\vec{D}=\rho$$
$$\vec{\nabla}.\vec{B}=0$$
$$\vec{\nabla}.\vec{J}=-\frac{\partial\rho}{\partial t}$$

$\Rightarrow$

$$\vec{\nabla}\times\vec{E}=-jw\vec{B}$$
$$\vec{\nabla}\times\vec{H}=\vec{J}+jw\vec{D}$$
$$\vec{\nabla}.\vec{D}=\rho$$
$$\vec{\nabla}.\vec{B}=0$$
$$\vec{\nabla}.\vec{J}=-jw\rho$$

$\vec{D}=\varepsilon\,\vec{E}$ $\quad\vec{B}=\mu\,\vec{H}$ $\quad\vec{J}=\sigma\,\vec{E}$

Plane wave in phasor form

Source free non-conductive medium:

$$\nabla^2\vec{E}(r,t)-\mu\,\varepsilon\frac{\partial^2\vec{E}(r,t)}{\partial t^2}=0$$

$$\vec{E}(r,t)=\vec{E}(r)e^{jwt}$$

$$\frac{\partial}{\partial t}\vec{E}(r,t)=\frac{\partial}{\partial t}\vec{E}(r)e^{jwt}=jw\vec{E}(r)e^{jwt}=jw\vec{E}(r,t)$$

$$\nabla^2\vec{E}+w^2\mu\,\varepsilon\,\vec{E}=0 \qquad \nabla^2\vec{H}+w^2\mu\,\varepsilon\,\vec{H}=0$$

Frequency analysis appearing

Plane wave in phasor form

$$\nabla^2\vec{E}+w^2\mu\,\varepsilon\,\vec{E}=0$$

$$\boxed{\nabla^2\vec{E}+\mathrm{k}^2\vec{E}=0}\quad\text{with}\quad \mathrm{k}^2=w^2\mu\,\varepsilon$$

Helmholtz equation

$$\frac{\partial^2E_x}{\partial z^2}+\mathrm{k}^2E_x=0 \qquad \frac{\partial^2H_x}{\partial z^2}+\mathrm{k}^2H_x=0$$

$$\frac{\partial^2E_y}{\partial z^2}+\mathrm{k}^2E_y=0 \qquad \frac{\partial^2H_y}{\partial z^2}+\mathrm{k}^2H_y=0$$

Plane wave in phasor form

Solution?

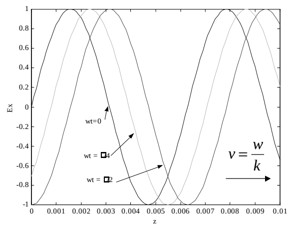
$$E_x=C_1e^{-jkz}+C_2e^{jkz}$$

$$\begin{aligned}E_x(z,t)&=\mathrm{Re}\left(E_x(z)e^{jwt}\right)\\&=\mathrm{Re}\left(C_1e^{j(wt-kz)}+C_2e^{j(wt+kz)}\right)\\&=\mathrm{Re}\left(C_1e^{jw(t-\frac{k}{w}z)}+C_2e^{jw(t+\frac{k}{w}z)}\right)\\&=Af(t-\frac{z}{v})+Bf(t+\frac{z}{v})\end{aligned}$$

Wave propagating in z and -z directions

Plane wave in phasor form

$$f(t-\frac{z}{v})=f(t+\delta\,\mathrm{t}-\frac{z+\delta\,\mathrm{z}}{v}) \Rightarrow \frac{\delta\,\mathrm{z}}{\delta\,\mathrm{t}}=v=\frac{w}{k} \Rightarrow k=\frac{w}{v}$$

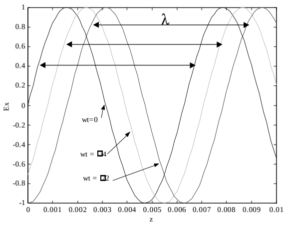


Phase velocity

$$E(z,t)=Ae^{jw(t-\frac{k}{w}z)}$$

Plane wave in phasor form

Wavelength λ:
- independent of time and position in space
- Characteristic of a wave



$$E(z,t) = Ae^{jw(t-\frac{k}{w}z)}$$

$$E(z+\lambda,t) = Ae^{jw(t-\frac{k}{w}(z+\lambda))} = Ae^{jw(t-\frac{k}{w}z)}$$

⇩

$$k\lambda = 2\pi$$

$$\frac{w}{v}\lambda = 2\pi$$

$$w\lambda = 2\pi \text{ v}$$

$$w = 2\pi \text{ f} \longrightarrow \text{v} = \text{f}\lambda$$

Plane wave in phasor form

$$k\lambda = 2\pi$$

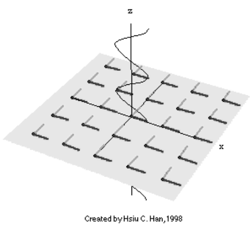
$$\frac{w}{v}\lambda = 2\pi$$

$$w\lambda = 2\pi \text{ v}$$

$$w = 2\pi \text{ f} \longrightarrow \text{v} = \text{f}\lambda \longleftarrow \begin{cases} \text{k} = \text{w}\sqrt{\mu\varepsilon} & \text{Wave number} \\ \lambda = \frac{2\pi}{k} & \text{Wavelength} \end{cases}$$

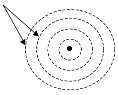
$$\text{v} = \frac{1}{\sqrt{\mu\,\varepsilon}} = \frac{1}{\sqrt{\mu_r\,\varepsilon_r}}\frac{1}{\sqrt{\mu_0\,\varepsilon_0}} = \frac{\text{c}}{\sqrt{\mu_r\,\varepsilon_r}}$$ Phase velocity

When do we have plane waves?



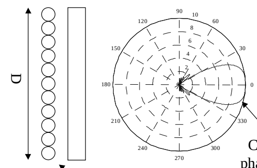
When do we have plane waves?

Constant phase fronts



Point source

Plane wave when: $r >= \frac{2D^2}{\lambda}$

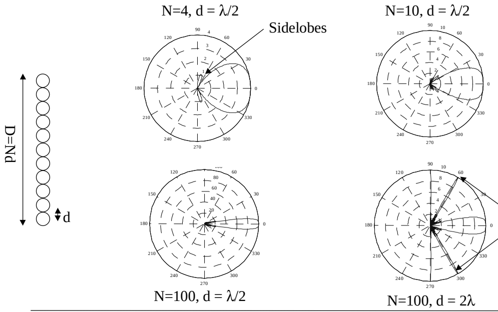


Constant phase fronts

Multiple small receivers (sampling) equivalent to one large antenna

Example: 1 meter antenna and f = 10 GHz
λ = 3 cm and rmin = 66.6 meters

Antennas



Power Flow

$$\vec{S}(r,t) = \vec{E}(r,t) \times \vec{H}(r,t) \quad \text{Poynting vector}$$

Waves are now sinusoidal and the instantaneous power is less useful

Electric analogy

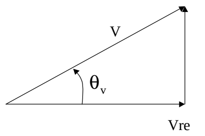
$$W = \hat{V}\hat{I}$$

$$\hat{V} = \text{Re}(Ve^{jwt}) = \text{Re}(|V|e^{j\theta_v}e^{jwt}) = |V|\cos(wt + \theta_v)$$

$$\hat{I} = \text{Re}(Ie^{jwt}) = \text{Re}(|I|e^{j\theta_i}e^{jwt}) = |I|\cos(wt + \theta_i)$$

Power Flow

$$W = \hat{V}\hat{I} = |V|\cos(wt+\theta_v)|I|\cos(wt+\theta_i)$$
$$= \frac{|V||I|}{2}\left[\underbrace{\cos(2wt+\theta_v+\theta_i)}_{\text{Null average over a period}}+\cos(\theta_v-\theta_i)\right]$$



$$W_{av} = \frac{|V||I|}{2}\cos(\theta_v-\theta_i) = \frac{|V||I|}{2}\cos\theta$$

Power Flow

Electricity

$$W_{react} = \frac{|V||I|}{2}\sin\theta$$
$$W = \frac{1}{2}VI^* = \frac{1}{2}|V||I|e^{j\theta}$$
$$= W_{av} + jW_{react}$$

Em Waves

$$\vec{p} = \frac{1}{2}\vec{E}\times\vec{H}^*$$

E and H are the phasors quantity

$$\vec{p}_{av} = \frac{1}{2}\mathrm{Re}(\vec{E}\times\vec{H}^*)$$
$$\vec{p}_{react} = \frac{1}{2}\mathrm{Im}(\vec{E}\times\vec{H}^*)$$

Power Flow

$$p_z = \frac{1}{2}(E_x H^*{}_y - E_y H^*{}_x)$$

But $\dfrac{E_x}{H_y} = -\dfrac{E_y}{H_x} = \eta = \sqrt{\dfrac{\mu_0}{\varepsilon}}$

$$H_x = -\sqrt{\frac{\mu_0}{\varepsilon}}E_y$$
$$H_y = \sqrt{\frac{\mu_0}{\varepsilon}}E_x$$

$$H^*{}_x = -\sqrt{\frac{\mu_0}{\varepsilon}}|E_y|e^{jkz}$$
$$H^*{}_y = \sqrt{\frac{\mu_0}{\varepsilon}}|E_x|e^{jkz}$$

$$p_z = \frac{1}{2}\left[\sqrt{\frac{\varepsilon}{\mu_0}}|E_x|^2 + \sqrt{\frac{\varepsilon}{\mu_0}}|E_y|^2\right]$$
$$= \frac{1}{2}v\varepsilon|E|^2$$

Power Flow

$$p_z = \frac{1}{2}v\varepsilon|E|^2$$

Electric energy density in the wave? $\dfrac{1}{2}\dfrac{\varepsilon|E|^2}{2} = \dfrac{\varepsilon|E|^2}{4}$ (See notes EM2)

Magnetic energy density in the wave? $\dfrac{1}{2}\dfrac{\mu|H|^2}{2} = \dfrac{\mu|H|^2}{4}$ (See notes EM2)

Total energy density? $\dfrac{\varepsilon|E|^2}{4} + \dfrac{\mu|H|^2}{4} = \dfrac{\varepsilon|E|^2}{4} + \dfrac{\mu}{4}\left[\sqrt{\dfrac{\varepsilon}{\mu_0}}E\right]^2 = \dfrac{1}{2}\varepsilon|E|^2$

The power flow in a plane wave is just the average energy density times the wave velocity

Propagation in conductive media

General wave equation for conducting medium

$$\nabla^2\vec{E}(r,t) - \mu_0\,\varepsilon\frac{\partial^2\vec{E}(r,t)}{\partial t^2} - \sigma\,\mu_0\frac{\partial\vec{E}(r,t)}{\partial t} = 0$$
$$\nabla^2\vec{H}(r,t) - \mu_0\,\varepsilon\frac{\partial^2\vec{H}(r,t)}{\partial t^2} - \sigma\,\mu_0\frac{\partial\vec{H}(r,t)}{\partial t} = 0$$

We assume a motion along the z axis

$$\frac{\partial^2\vec{E}(r,t)}{\partial z^2} - \mu_0\,\varepsilon\frac{\partial^2\vec{E}(r,t)}{\partial t^2} - \sigma\,\mu_0\frac{\partial\vec{E}(r,t)}{\partial t} = 0$$
$$\frac{\partial^2\vec{H}(r,t)}{\partial z^2} - \mu_0\,\varepsilon\frac{\partial^2\vec{H}(r,t)}{\partial t^2} - \sigma\,\mu_0\frac{\partial\vec{H}(r,t)}{\partial t} = 0$$

Propagation in conductive media

In phasor form it becomes

$$\frac{\partial^2\vec{E}}{\partial z^2} + \mu_0\,\varepsilon w^2\vec{E} - j\sigma\,\mu_0 w\vec{E} = 0$$
$$\frac{\partial^2\vec{E}}{\partial z^2} + (\mu_0\,\varepsilon w^2 - j\sigma\,\mu_0 w)\vec{E} = 0$$

Helmholtz form

$$\frac{\partial^2\vec{E}}{\partial z^2} - \gamma^2\,\vec{E} = 0$$

where $\gamma^2 = jw\mu_0(\sigma + jw\varepsilon)$

$\gamma = \alpha + j\beta$ With $\alpha > 0$

Solution? $\vec{E}(z) = \vec{E}_0 e^{-\gamma z}$

What if $\alpha < 0$?

Propagation in conductive media

$\gamma = \alpha + j\beta$

$\vec{E}(z) = \vec{E}_0 e^{-\gamma z}$

$$\vec{E}(z,t) = \mathrm{Re}\left(\vec{E}_0 e^{(-\gamma z + jwt)}\right) = e^{-\alpha z}\,\mathrm{Re}\left(\vec{E}_0 e^{(jwt - \beta z)}\right)$$

Travelling wave in the z direction attenuated by a factor $e^{-\alpha z}$

As in the loss-less case, the phase velocity is given by: $v = \dfrac{w}{\beta}$ with $\beta = \dfrac{2\pi}{\lambda}$

Propagation in conductive media

$\alpha = \mathrm{Re}\left\{\left[\mathrm{j}w\mu_0\,(\sigma + \mathrm{j}w\varepsilon)\right]^{\frac{1}{2}}\right\}$

$\alpha = w\sqrt{\dfrac{\mu_0\,\varepsilon}{2}\left\{\sqrt{(1+\dfrac{\sigma^2}{w^2\varepsilon^2})} - 1\right\}}$ Exact formula!

$\beta = \mathrm{Im}\left\{\left[\mathrm{j}w\mu_0\,(\sigma + \mathrm{j}w\varepsilon)\right]^{\frac{1}{2}}\right\}$

$\beta = w\sqrt{\dfrac{\mu_0\,\varepsilon}{2}\left\{\sqrt{(1+\dfrac{\sigma^2}{w^2\varepsilon^2})} + 1\right\}}$ Exact formula!

Good Dielectric case: $\dfrac{\sigma}{w\varepsilon} << 1$

$\dfrac{\sigma}{w\varepsilon}$ is called the dissipation factor

Example: Mica at radio frequencies $\dfrac{\sigma}{w\varepsilon} = 0.0002$

$\sqrt{(1+\dfrac{\sigma^2}{w^2\varepsilon^2})} = 1 + \dfrac{\sigma^2}{2w^2\varepsilon^2} + o(\dfrac{\sigma^2}{w^2\varepsilon^2})$ (Taylor expansion)

$\alpha = w\sqrt{\dfrac{\mu_0\,\varepsilon}{2}\left\{(1+\dfrac{\sigma^2}{2w^2\varepsilon^2}) - 1\right\}} = w\sqrt{\dfrac{\mu_0\,\varepsilon}{2}\left\{\dfrac{\sigma^2}{2w^2\varepsilon^2}\right\}}$

$= \dfrac{\sigma}{2}\sqrt{\dfrac{\mu_0}{\varepsilon}} = \dfrac{\sigma}{2}\sqrt{\dfrac{\mu_0}{\varepsilon_0}}\sqrt{\dfrac{1}{\varepsilon_r}} = \dfrac{\sigma}{2}\eta_0\sqrt{\dfrac{1}{\varepsilon_r}}$ $[\eta_0 = 377\Omega]$

Good Dielectric case: $\dfrac{\sigma}{w\varepsilon} << 1$

$\beta = w\sqrt{\dfrac{\mu_0\,\varepsilon}{2}\left\{(1+\dfrac{\sigma^2}{2w^2\varepsilon^2}) + 1\right\}} = w\sqrt{\mu_0\,\varepsilon\left\{1+\dfrac{\sigma^2}{4w^2\varepsilon^2}\right\}}$

$= w\sqrt{\mu_0\,\varepsilon}\left(1+\dfrac{\sigma^2}{8w^2\varepsilon^2}\right)$

Perfect dielectric — Correction term

Wave velocity $v = \dfrac{w}{\beta} = \dfrac{1}{\sqrt{\mu_0\,\varepsilon}\left(1+\dfrac{\sigma^2}{8w^2\varepsilon^2}\right)}$

$= v_0\left(1 - \dfrac{\sigma^2}{8w^2\varepsilon^2}\right)$

Good Conductor case: $\dfrac{\sigma}{w\varepsilon} >> 1$

Even at microwave range (~30 GHz) $\dfrac{\sigma}{w\varepsilon} \approx 3.5\ 10^8$ for Copper

When $\dfrac{\sigma}{w\varepsilon} >> 1$ we can write

$\gamma = \sqrt{\mathrm{j}w\mu_0\,(\sigma + \mathrm{j}w\varepsilon)} = \sqrt{\mathrm{j}w\mu_0\,\sigma} = \sqrt{w\mu_0\,\sigma}\angle 45^0$

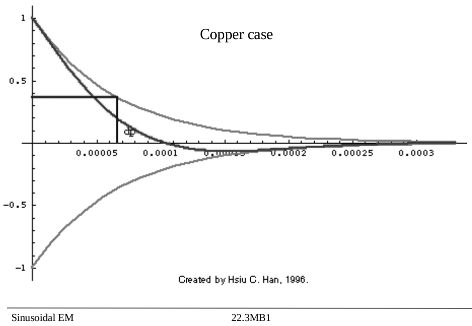
$\gamma = \alpha + \mathrm{j}\beta = \sqrt{\dfrac{w\mu_0\,\sigma}{2}} + \mathrm{j}\sqrt{\dfrac{w\mu_0\,\sigma}{2}}$

$\alpha = \beta = \sqrt{\dfrac{w\mu_0\,\sigma}{2}}$

Wave velocity

$v = \dfrac{w}{\beta} = \sqrt{\dfrac{2w}{\mu_0\sigma}}$

Good Conductor case: $\dfrac{\sigma}{w\varepsilon} >> 1$



Wave impedance

Wave Impedance

$$\eta = \frac{E}{H} \qquad \nabla x \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$\frac{\partial E_x}{\partial z} = -jw\mu_0 H_y \qquad -\gamma E_x = -jw\mu_0 H_y$$

$$\frac{E_x}{H_y} = \eta = \frac{jw\mu_0}{\gamma} \qquad \eta = \frac{jw\mu_0}{(\alpha + j\beta)}$$

Good Dielectric case:

$$\eta = \frac{jw\mu_0}{(\alpha + j\beta)} = \frac{jw\mu_0}{\sqrt{jw\mu_0\,(\sigma + jw\varepsilon)}} = \sqrt{\frac{jw\mu_0}{(\sigma + jw\varepsilon)}}$$

Good dielectric: $\frac{\sigma}{w\varepsilon} << 1$

$$\eta = \sqrt{\frac{jw\mu_0}{(\sigma + jw\varepsilon)}} = \sqrt{\frac{\mu_0}{\varepsilon}\frac{1}{(1+\frac{\sigma}{jw\varepsilon})}} = \sqrt{\frac{\mu_0}{\varepsilon}}(1+\frac{j\sigma}{2w\varepsilon})$$

Intrinsic impedance

Small reactive component

Good conductor case:

$$\eta = \frac{jw\mu_0}{(\alpha + j\beta)} = \frac{jw\mu_0}{\sqrt{jw\mu_0\,(\sigma + jw\varepsilon)}} = \sqrt{\frac{jw\mu_0}{(\sigma + jw\varepsilon)}}$$

Good conductor: $\frac{\sigma}{w\varepsilon} >> 1$

$$\eta = \sqrt{\frac{jw\mu_0}{\sigma}} = \sqrt{\frac{w\mu_0}{\sigma}}\angle 45^0$$

Skin Depth σ not negligable

$$\vec{E}(z,t) = e^{-\alpha z}\,\mathrm{Re}\left(\vec{E}_0 e^{(jwt-\beta z)}\right) \qquad \alpha = w\sqrt{\frac{\mu_0\,\varepsilon}{2}\left\{\sqrt{(1+\frac{\sigma^2}{w^2\varepsilon^2})}-1\right\}}$$

attenuation

Definition: the skin depth δ is defined as the distance in the direction of propagation in which the wave has been attenuated by $\frac{1}{e} = 37\%$

$$\delta = \frac{1}{\alpha} = \frac{1}{w\sqrt{\frac{\mu_0\,\varepsilon}{2}\left\{\sqrt{(1+\frac{\sigma^2}{w^2\varepsilon^2})}-1\right\}}}$$

Good conductor

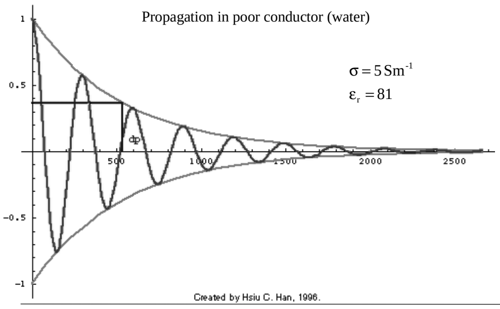
$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\sigma\mu_0 w}}$$

Skin Depth σ not negligable

Propagation in poor conductor (water)

$\sigma = 5\,Sm^{-1}$

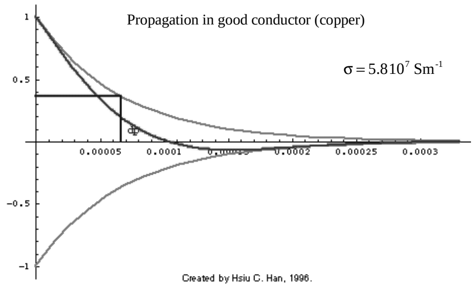
$\varepsilon_r = 81$



Skin Depth σ not negligable

Propagation in good conductor (copper)

$\sigma = 5.8 10^7\ Sm^{-1}$



Skin Depth

σ not negligable

Propagation in good conductor (copper)

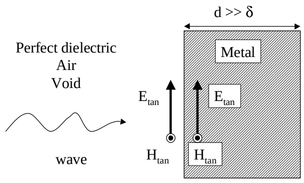
$\sigma = 5.810^{7}\ \mathrm{Sm^{-1}} \qquad f = 1\mathrm{Mhz} \qquad \mu = \mu_0 = 4\pi 10^{-7}\,\mathrm{H}$

Good conductor
$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\sigma\mu_0 w}}$

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{5.810^{7}\, 4\pi 10^{-7}\, 2\pi 10^{6}}} = 0.0667\mathrm{mm}$$

Power Loss in Metal

We have seen (boundary conditions)



$$\frac{E_{tan}}{H_{tan}} = \eta_{metal}$$

$$\eta_{metal} = \sqrt{\frac{jw\mu_0}{(\sigma + jw\varepsilon)}}$$

$$\eta_{metal} = \sqrt{\frac{jw\mu_0}{\sigma}} = \sqrt{\frac{w\mu_0}{\sigma}}\angle 45^0$$

$$p_{av} = \frac{1}{2}\mathrm{Re}(E_{tan} \times H_{tan}^{*})$$

Power Loss in Metal

$$\frac{E_{tan}}{H_{tan}} = \eta_{metal} \qquad \eta_{metal} = \sqrt{\frac{jw\mu_0}{\sigma}} = \sqrt{\frac{w\mu_0}{\sigma}}\angle 45^0$$

$$p_{av} = \frac{1}{2}\mathrm{Re}(E_{tan} \times H_{tan}^{*}) = \frac{1}{2}|E_{tan}||H_{tan}|\cos(\pi/4)$$

$$= \frac{1}{2\sqrt{2}}\eta_{metal}|H_{tan}|^2$$

$$= \frac{1}{2\sqrt{2}}\frac{|E_{tan}|^2}{\eta_{metal}}$$