HERIOT-WATT UNIVERSITY
DEPARTMENT OF COMPUTING AND ELECTRICAL ENGINEERING

22.3MB1 Electromagnetics

Tutorial 2 Correction

1. Determine the gradient of the following potential: $V = x^2y + zy^2 + xyz$

$$[\ \overrightarrow{\nabla V} = (2xy + yz)\vec{a}_x + (x^2 + 2yz + xz)\vec{a}_y + (y^2 + xy)\vec{a}_z\ ]$$

2. Determine the divergence of the following vector: $\vec{E} = y^2\vec{a}_x + z^2\vec{a}_y + x^2\vec{a}_z$

$$[\ \nabla.E = 0\ ]$$

3. Determine the divergence of the following vector: $\vec{E} = x\ y\vec{a}_x + yz\ \vec{a}_y + zx\ \vec{a}_z$

$$[\ \nabla.E = y + z + x\ ]$$

4. Calculate the curl of the following vector: $E = y^2\vec{a}_x + z^2\vec{a}_y + x^2\vec{a}_z$

$$[\ \nabla x\vec{E} = -2(z\vec{a}_x + x\vec{a}_y + y\vec{a}_z)\ ]$$

5. Show that, in cartesian coordinates

$$\nabla \times \nabla \times \underline{A} = \nabla\nabla \cdot \underline{A} - \nabla^2 \underline{A}$$

We could use the cartesian coordinates and show that the result holds. This involves caluclating the first curl and then applying curl again to the result. It is a lengthy calculation and is left to the student. However, there is a much faster and elegant way to derive the result, which is also independent of the coordinates system.
If you remember the following vector algebra formula:
$\vec{A}\times\vec{B}\times\vec{C} = \vec{A}(\vec{B}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$ (1)
and apply it to our case you find:
$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{A}(\vec{\nabla}.\vec{\nabla})$
The first term is obviously what we are looking for but the second does not make any sense. This is because we are dealing with operators and have not been careful enough when using formula (1). It can be rewritten as :
$\vec{A}\times\vec{B}\times\vec{C} = \vec{A}(\vec{B}.\vec{C}) - (\vec{A}.\vec{B})\vec{C}$ (2)
It means the same thing for vectors but does actually also make sense for operators as proven below:
$\vec{\nabla}\times\vec{\nabla}\times\vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - (\vec{\nabla}.\vec{\nabla})\vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - (\nabla^2)\vec{A}$
where ∇^2 is the laplacian operator defined as:
$\nabla^2 = \dfrac{\partial^2}{\partial^2 x^2} + \dfrac{\partial^2}{\partial^2 y^2} + \dfrac{\partial^2}{\partial^2 z^2}$ and can be applied to vectors as:

$$\nabla^2\vec{A} = \begin{bmatrix} \nabla^2 A_x \\ \nabla^2 A_y \\ \nabla^2 A_z \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 A_x}{\partial^2 x^2} + \frac{\partial^2 A_x}{\partial^2 y^2} + \frac{\partial^2 A_x}{\partial^2 z^2} \\ \frac{\partial^2 A_y}{\partial^2 x^2} + \frac{\partial^2 A_y}{\partial^2 y^2} + \frac{\partial^2 A_y}{\partial^2 z^2} \\ \frac{\partial^2 A_z}{\partial^2 x^2} + \frac{\partial^2 A_z}{\partial^2 y^2} + \frac{\partial^2 A_z}{\partial^2 z^2} \end{bmatrix}$$

6. Show that, in a charge free, current free dielectric, Maxwell's two divergence equations may be derived from the two curl equations so far as time-varying parts of the field are concerned.

Lets remind ourselves of the Maxwell equations in differential form in their most general expression first:

$$\mathrm{div}(\vec{B}) = \vec{\nabla}.\vec{B} = 0$$

$$\mathrm{div}(\vec{D}) = \vec{\nabla}.\vec{D} = \rho$$

$$\mathrm{curl}(\vec{H}) = \vec{\nabla}x\vec{H} = \vec{j} + \frac{\partial\vec{D}}{\partial t}$$

$$\mathrm{curl}(\vec{E}) = \vec{\nabla}x\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

In our case, we know that $\vec{j}$ and ρ are null. Thefore we can write:

$$\mathrm{div}(\vec{B}) = \vec{\nabla}.\vec{B} = 0$$

$$\mathrm{div}(\vec{D}) = \vec{\nabla}.\vec{D} = 0$$

$$\mathrm{curl}(\vec{H}) = \vec{\nabla}x\vec{H} = \frac{\partial\vec{D}}{\partial t}$$

$$\mathrm{curl}(\vec{E}) = \vec{\nabla}x\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Starting with the last 2 equations and using the fact that $\vec{\nabla}.(\vec{\nabla}x\vec{A}) = 0$ for any vector A (see notes) we can deduce:

$$\mathrm{div}(\mathrm{curl}(\vec{H})) = \vec{\nabla}.(\vec{\nabla}x\vec{H}) = 0 = \mathrm{div}(\frac{\partial\vec{D}}{\partial t}) = \frac{\partial \mathrm{div}(\vec{D})}{\partial t}$$

$$\mathrm{div}(\mathrm{curl}(\vec{E})) = \vec{\nabla}.(\vec{\nabla}x\vec{E}) = 0 = \mathrm{div}(-\frac{\partial\vec{B}}{\partial t}) = -\frac{\partial \mathrm{div}(\vec{B})}{\partial t}$$

Therefore we have proven the divergence equations from the curl ones for the time varying part of the field.

7. Show that, if the equation for continuity of charge is assumed, the two divergence relations in Maxwell's equations may be derived from the curl equations so far as a-c components of the field are concerned, for regions with finite ρ and J̲.

Here, we are going to use the same argument but we need to be able to relate $\vec{j}$ and ρ using the continuity of charge assumption as:

$$\mathrm{div}(\vec{j}) = \vec{\nabla}.\vec{j} = -\frac{\partial \rho}{\partial t}$$

We start again from the maxwell equations:

$$\mathrm{div}(\vec{B}) = \vec{\nabla}.\vec{B} = 0$$

$$\mathrm{div}(\vec{D}) = \vec{\nabla}.\vec{D} = \rho$$

$$\mathrm{curl}(\vec{H}) = \vec{\nabla}\mathrm{x}\vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

$$\mathrm{curl}(\vec{E}) = \vec{\nabla}\mathrm{x}\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

we then have (as in 6):

$$\mathrm{div}(\mathrm{curl}(\vec{H})) = \vec{\nabla}.(\vec{\nabla}\mathrm{x}\vec{H}) = 0 = \mathrm{div}(\vec{j} + \frac{\partial \vec{D}}{\partial t}) = \mathrm{div}(\vec{j}) + \frac{\partial}{\partial t}\mathrm{div}(\vec{D}) = -\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial t}\mathrm{div}(\vec{D})$$

$$\mathrm{div}(\mathrm{curl}(\vec{E})) = \vec{\nabla}.(\vec{\nabla}\mathrm{x}\vec{E}) = 0 = \mathrm{div}(-\frac{\partial \vec{B}}{\partial t}) = -\frac{\partial \mathrm{div}(\vec{B})}{\partial t}$$

Which can be simplified as

$$\frac{\partial}{\partial t}(\mathrm{div}(\vec{D}) - \rho) = 0$$

$$-\frac{\partial}{\partial t}(\mathrm{div}(\vec{B})) = 0$$

Which proves the divergence equations from the curl ones again fro the time varying parts of the field.