Electromagnetics: Displacement current, Divergence and Stokes Theorem, Gradients

22.2MB1

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Section Contents

- Displacement current and EM
- Maxwell equation in differential form:
  - The nabla operator
  - Gradient of a potential field
  - Divergence of a field
  - Divergence theorem
  - Curl of a field
  - Stokes Theorem
- Vector Calculus memo
- A word on simulation tools

EM Fields fundamentals

$\oint_C \vec{E}.d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}.d\vec{S}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
$\oint_C \vec{H}.d\vec{l} = \int_S \left(J + \frac{\partial \vec{D}}{\partial t} \right) d\vec{S}$	$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$
$\oint_S \vec{D}.d\vec{S} = \int_v \rho dv$	$\vec{\nabla}.\vec{D} = \rho$
$\oint_S \vec{B}.d\vec{S} = 0$	$\vec{\nabla}.\vec{B} = 0$

Inconsistencies of Ampere's Law

$\oint_C \vec{H}.d\vec{l} = \int_S \left(J + \frac{\partial \vec{D}}{\partial t} \right) d\vec{S}$ Maxwell's 2nd Equation = Ampere's Law

What is this ?

Let's start from the beginning...

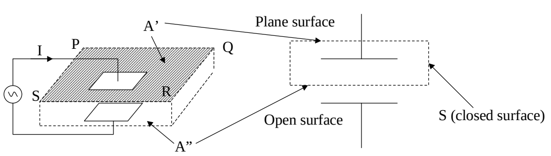
$$\iint \vec{J}.\overrightarrow{dS} = -\frac{\partial}{\partial t} \iiint \rho dV$$

Conservation of the charge for varying fields

Charges Cannot be created or destroyed

J : Current flowing out of some volume

Inconsistencies of Ampere's Law



Ampere's Law

$$\oint_C \vec{H}.d\vec{l} = \int_{A'} \vec{J}.d\vec{A}'$$

Ampere's Law

$$\oint_C \vec{H}.d\vec{l} = \int_{A''} \vec{J}.d\vec{A}'' = 0$$

Problem!

Displacement current

We introduce a displacement current as:

$$\oint_C \vec{H}.d\vec{l} = \int_{A''} \vec{J}_d.d\vec{A}'' = I_d$$

We now have

$$\int (\vec{J}_d + \vec{J}).dS = 0 \quad (1)$$

We also have

$$\iint \vec{J}.\overrightarrow{dS} = -\frac{\partial}{\partial t} \iiint \rho dV$$

$$\oint_S \vec{D}.d\vec{S} = \int_v \rho dV$$

$\Rightarrow$ $\int (\vec{J} + \frac{\partial \vec{D}}{\partial t}).\overrightarrow{dS} = 0 \quad (2)$

$\vec{J}$ and $\frac{\partial \vec{D}}{\partial t}$ cannot be equalized

Displacement current

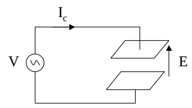
(1) and (2) gives:

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$$

Therefore:

$$\oint_C \vec{H}.d\vec{l} = \int_S \left(J + \frac{\partial \vec{D}}{\partial t} \right) d\vec{S}$$ Maxwell's 2nd Equation for fields

Application example



$V = V_0 \cos(wt)$

1) Find I_c using standard Circuits theory

2) Find I_d using EM theory

Comments ?

Maxwell equations integral form

$\oint_C \vec{E}.d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}.d\vec{S}$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
$\oint_C \vec{H}.d\vec{l} = \int_S \left(J + \frac{\partial \vec{D}}{\partial t} \right) d\vec{S}$	$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$
$\oint_S \vec{D}.d\vec{S} = \int_v \rho dv$	$\vec{\nabla}.\vec{D} = \rho$
$\oint_S \vec{B}.d\vec{S} = 0$	$\vec{\nabla}.\vec{B} = 0$
$\oint_S \vec{J}.d\vec{S} = -\frac{\partial}{\partial t} \iiint \rho dV$	$\vec{\nabla}.\vec{J} = -\frac{\partial \rho}{\partial t}$

Why Differential form?

- Time Varying fields
- 3D problems
- Needs the determination of fields in specific points in space
- Not very easy to use as a mathematical tool

- Use Differential form
- Nabla Operator
- Divergence
- Gradient
- Curl

Gradient o f a potential field

$$V = -\int_l \vec{E}.d\vec{l} \quad (\text{Volts})$$ Potential field

$$dV = -\vec{E}.d\vec{l}$$ with $$\vec{dl} = \vec{a}_x d_x + \vec{a}_y d_y + \vec{a}_z d_z$$ $$\vec{E} = \vec{a}_x E_x + \vec{a}_y E_y + \vec{a}_z E_z$$

$$dV = -(E_x dx + E_y dy + E_z dz)$$

$$dV = \left(\frac{\partial V(x,y,z)}{\partial x} dx + \frac{\partial V(x,y,z)}{\partial y} dy + \frac{\partial V(x,y,z)}{\partial z} dz\right)$$

$$E_x = -\frac{\partial V(x,y,z)}{\partial x}$$

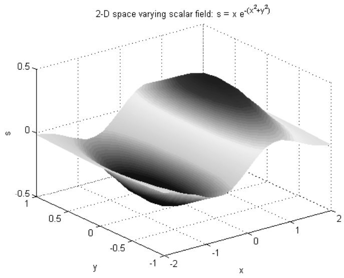
$$E_y = -\frac{\partial V(x,y,z)}{\partial y}$$

$$E_z = -\frac{\partial V(x,y,z)}{\partial z}$$

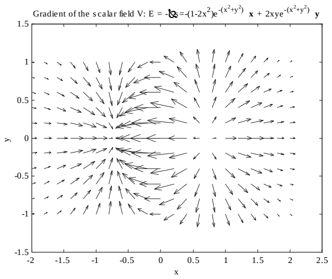
$$\vec{E} = -\left(\vec{a}_x \frac{\partial V}{\partial x} + \vec{a}_y \frac{\partial V}{\partial y} + \vec{a}_z \frac{\partial V}{\partial z}\right)$$

$$\vec{E} = -\overrightarrow{\text{grad}}\ V$$

Gradient of a potential: Example



Gradient of a potential: Example



Gradient of a potential: Nabla operator

$(\vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z})$ Vector operator written: $\vec{\nabla}$ or ∇

$$\vec{\nabla} V = (\vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z})V$$
$$= \vec{a}_x \frac{\partial V}{\partial x} + \vec{a}_y \frac{\partial V}{\partial y} + \vec{a}_z \frac{\partial V}{\partial z}$$

Vector

$$\vec{E} = -\overrightarrow{\text{grad}}\ V = \vec{\nabla} V$$

Gradient of a potential: Example

Consider the following potential:

$$V = \frac{1}{xyz} e^{-(x^2+yz)}$$

Calculate the gradient and derive the field E:

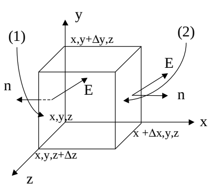
Divergence of a Vector Field

$$\iint \vec{D}.\overrightarrow{dS} = \iiint \rho dV$$ Gauss Law

$$\lim_{\Delta V \to 0} \frac{\iint \vec{D}.\overrightarrow{dS}}{\Delta V} = \frac{\iiint \rho dV}{\Delta V} \ ?$$

$$\text{div}\vec{D} = \rho$$

Divergence of a Vector Field



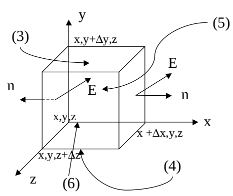
Flux out of $1 = \Phi_1 = -\int \vec{E}_x(1)dxdz$

Flux out of $2 = \Phi_2 = -\int \vec{E}_x(2)dxdz$

But $E_x(2) = E_x(x+\Delta x) = E_x(x) + \frac{\partial E_x}{\partial x}dx = E_x(1) + \frac{\partial E_x}{\partial x}dx$

Therefore $\Phi_1 + \Phi_2 = \int \frac{\partial E_x}{\partial x} dxdydz$

Divergence of a Vector Field



$\Phi_3 + \Phi_4 = \int \frac{\partial E_y}{\partial y} dxdydz$

$\Phi_5 + \Phi_6 = \int \frac{\partial E_z}{\partial z} dxdydz$

$\oint_S \vec{E}.d\vec{S} = \iiint_V \text{div}(\vec{E}).dV$

$\text{div}(\vec{E}) = (\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z})$ Scalar

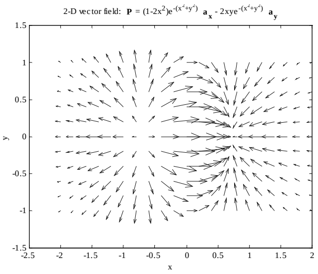
Divergence of a Vector Field

$$\mathrm{div}(\vec{E}) = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) \qquad \underline{\text{Scalar}}$$

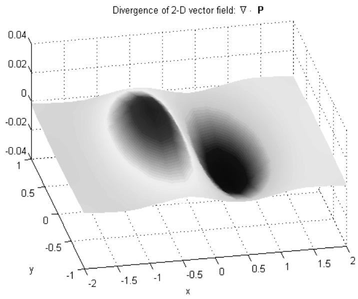
$$\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) = \left(\vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}\right).(\vec{a}_x E_x + \vec{a}_y E_y + \vec{a}_z E_z)$$

$$\mathrm{div}\vec{E} = \vec{\nabla}.\vec{E} \qquad \underline{\text{Scalar as dot product of two vectors}}$$

Example of divergence



Example of divergence



Divergence of a field: Example

Consider the following Field:

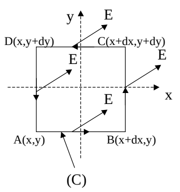
$$\vec{E} = (1 - 2x^2)e^{-(x^2+yz)}\vec{a}_x - 2xye^{-(x^2+yz)}\vec{a}_y$$

Calculate the divergence of the field E:

Curl of a Field: Stokes Theorem

Circulation of a field

$$C(\vec{E}) = \int \vec{E}.d\vec{l} = \int E_x dx + E_y dy + E_z dz$$



$$\oint_c \vec{E}.d\vec{l} = \int_A^B \vec{E}.d\vec{l} + \int_B^C \vec{E}.d\vec{l} + \int_C^D \vec{E}.d\vec{l} + \int_D^A \vec{E}.d\vec{l}$$

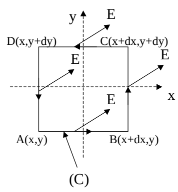
$$\int_A^B \vec{E}.d\vec{l} = \int_A^B E_x(x,y)dx \qquad \int_D^A \vec{E}.d\vec{l} = -\int_D^A E_y(x,y)dx$$

$$\int_B^C \vec{E}.d\vec{l} = \int_B^C E_y(x+dx,y)dy = \int_B^C \left[E_y(x,y) + \frac{\partial E_y}{\partial x}dx\right]dy$$

$$\int_C^D \vec{E}.d\vec{l} = \int_C^D -E_x(x,y+dy)dx = -\int_C^D \left[E_x(x,y) + \frac{\partial E_x}{\partial y}dy\right]dx$$

$$\oint_c \vec{E}.d\vec{l} = \iint \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)dxdy$$

Curl of a Field: Stokes Theorem



$$\oint_c \vec{E}.d\vec{l} = \iint \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)dxdy$$

$$\oint_c \vec{E}.d\vec{l} = \iint \overrightarrow{\mathrm{curl}(\vec{E})}.\overrightarrow{dS} \qquad \underline{\text{Stokes Theorem}}$$

Curl measures the rotational part of a field

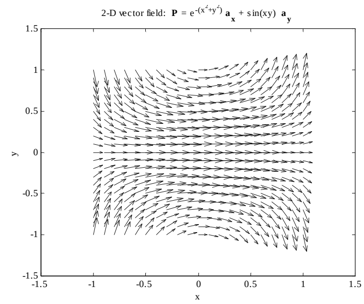
Curl of a Field: Nabla operator

$$\overrightarrow{\mathrm{curl}}(\vec{E}) = \vec{a}_x\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \vec{a}_y\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \vec{a}_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

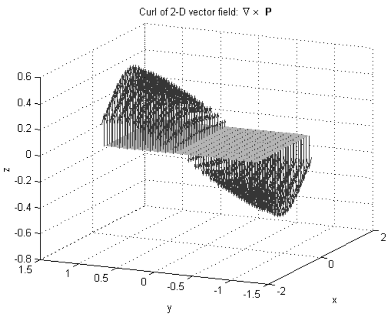
Vector

$$\overrightarrow{\mathrm{curl}}(\vec{E}) = \vec{\nabla} x \vec{E} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

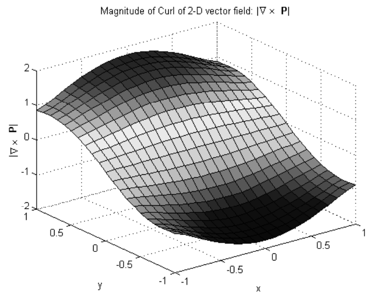
Curl of a Field: Example



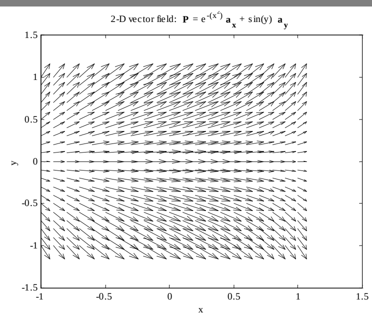
Curl of a Field: Example



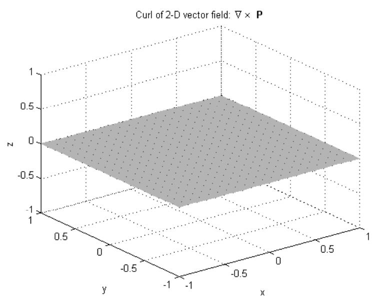
Curl of a Field: Example



Curl of a Field: Example 2



Curl of a Field: Example



Curl: Calculation example

Consider the following Field:

$$\vec{E} = e^{-(x^2+y^2)}\vec{a}_x + \sin(xy)\vec{a}_y$$

Calculate the curl of the field E:

Consider the following Field:

$$\vec{E} = e^{-(x^2+yz)}\vec{a}_x + \sin(z)\vec{a}_y$$

Calculate the curl of the field E:

Formulas memorandum

Gradient

$$\text{grad}(V) = \vec{\nabla}V = \vec{a}_x\frac{\partial V}{\partial x} + \vec{a}_y\frac{\partial V}{\partial y} + \vec{a}_z\frac{\partial V}{\partial z}$$

Vector field

Divergence

$$\text{div}(\vec{E}) = \vec{\nabla}.\vec{E} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)$$

Scalar field

Curl

$$\overrightarrow{\text{curl}}(\vec{E}) = \vec{\nabla}X\vec{E} = \vec{a}_x\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \vec{a}_y\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \vec{a}_z\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

Vector field

Formulas memorandum

$\text{div}(\text{curl}(E) = \nabla.(\nabla x E) = 0$	Scalar field
$\text{curl}(\text{grad}(E)) = \nabla X(\nabla E) = (\nabla X \nabla)E = 0$	Vector field
$\text{div}(\text{grad}(V)) = \nabla.(\nabla V) = \nabla^2 V$	Scalar field
$(\nabla.\nabla)E = \nabla^2 E$	Vector field
$\nabla(\nabla.E) = \text{a vector field}$	Scalar field
$\nabla X(\nabla X E) = \nabla(\nabla.E) - \nabla^2 E$	Vector field

Maxwell Equations: Differential form

1-Faraday Law

$$\oint_C \vec{E}.d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}.d\vec{S}$$

Stokes Theorem

$$\oint_c \vec{E}d\vec{l} = \iint \overrightarrow{\text{curl}}(\vec{E}).\overrightarrow{dS} = \iint (\vec{\nabla}, x\vec{E}).\overrightarrow{dS}$$

$$\Rightarrow \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Maxwell Equations: Differential form

1-Ampere's Law

$$\oint_C \vec{H}.d\vec{l} = \int_S \left(J + \frac{\partial \vec{D}}{\partial t}\right)d\vec{S}$$

Stokes Theorem

$$\Rightarrow \vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Maxwell Equations: Differential form

1-Gauss's Law

$$\oint_S \vec{D}.d\vec{S} = \int_v \rho dv$$

Divergence Theorem

$$\oint_S \vec{E}.d\vec{S} = \iiint_V \text{div}(\vec{E}).dV = \iiint_V (\vec{\nabla}.\vec{E}).dV$$

$$\Rightarrow \vec{\nabla}.\vec{D} = \rho$$

Maxwell Equations: Differential form

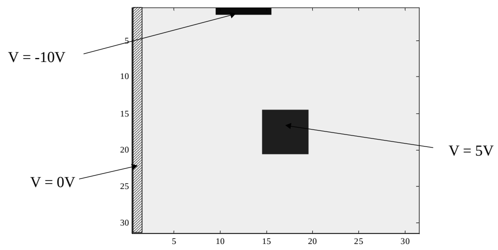
-4th Maxwell equation $\oint_S \vec{B}.d\vec{S} = 0$

Divergence Theorem ⇒ $\vec{\nabla}.\vec{B} = 0$

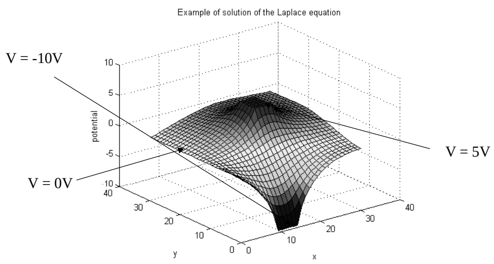
-Continuity equation $\iint \vec{J}.\overrightarrow{dS} = -\frac{\partial}{\partial t}\iiint \rho dV$

Divergence Theorem ⇒ $\vec{\nabla}.\vec{J} = -\frac{\partial \rho}{\partial t}$

Computational Electromagnetics: Example with Poisson equation



Computational Electromagnetics: Example with Poisson equation



Computational Electromagnetics: Example with Poisson equation