Electromagnetics Tutorial

1. plane wave is incident normally on a sheet of polystyrene of dielectric constant ε_r with a large hole. Derive an expression for how thick the sheet should be so that the wave that has passed through the sheet is in phase with the wave that has passed through the hole. For teflon with a value of $\varepsilon_r=2.7$ and a frequency of 10 GHz, how thick should the Teflon sheet be?

2. When a lens is used to form an image on the detector within a camera, reflections from the lens surfaces results in loss of transmitted light and stray light reflections at the detector. Estimate, stating any simplifying assumptions made, the fraction of the available optical power at the input surface that is transmitted directly to the detector (this forms the image) and the fraction that arrives at the detector via the most intense to reflection from the lens surfaces (this is background and ghost images) for (a) operation in the visible using glass lenses ($n =1.5$) and (b) in the thermal infrared using germanium lenses ($n=4$). You will need to derive the basic expressions based on reflectivity equations given in the lectures.

3. Reflections can be reduced by coating a lens with an antireflection coating consisting of a layer of dielectric that is $\lambda/4$ thick whereby that the reflection from the front surface is equal in amplitude, but opposite in phase with the reflection at the interface between the lens and coating. By assuming low refractive index and only the two strongest reflected components, show that if the lens has a refractive index of n, the refractive index of the coating material be $\sqrt{n}$. For a glass lens with $n=1.5$, what will be the thickness of the dielectric to reduce reflections for green light with a wavelength of 550 nm. Can you explain why the reflections from such a lens appear purple when illuminated by white light?

1. Explain why light from the sun tends to be unpolarised whilst radio waves from a dipole are linearly polarised.

2. Four linear polarisers are stacked together with mutual angular displacements of 30º, what is the fractional power transmission through all four when illuminated by natural light.

3. The intensity of solar radiation at the top of the Earth's atmosphere is 1.4 kW/m². Estimate the RMS strengths of the E-field and B-field of this radiation.

4. A short dipole is aligned vertically and a negative charge is applied to the whole dipole. Sketch the electric field lines close to the dipole and at a great distance (that is on a greatly expanded scale, so that the dipole appears as point source)

   A high frequency, radio-frequency sinusoidal power source is now applied to the centre of the dipole to produce a current that oscillates along the dipole. Sketch the electric field lines close to the dipole and at a great distance, for which the evanescent Coulomb-field is insignificant.

5. With reference to suitable diagrams show the instantaneous time and space relationship between electric field and magnetic field for a sinusoidal TEM (transverse electromagnetic) wave propagating in an unbounded region of free space.

Indicate the significance of the Poynting vector for the wave, both graphically and in vector notation

By considering a laser beam as a plane transverse electromagnetic (TEM) wave, estimate the diameter of the beam of a 1kW laser that would cause voltage breakdown in air. Assume that air has dielectric breakdown strength of 3MV/m

6. A parallel plate capacitor with circular plates of radius R and separation d is charged with a voltage ramp that increases linearly at α V/sec. Using Amperes equation in integral form, $\oint \mathbf{H.ds} = \iint_A \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) d\mathbf{A}$, derive an expression for the variation with radius r, of the magnetic field between the plates of the capacitor, for $r \leq R$.

Derive an expression for the magnetic field for $r > R$.

Derive an expression in terms of R, α and d for the current flowing through the wires that feed the capacitor.

Use this expression and Amperes equation to derive an expression for the magnetic field that this current produces around the wires.

Compare the expressions for the magnetic fields around the wire and around the capacitor. Is it possible, to distinguish between the magnetic field due to the displacement current and that due to the electron current?

If the voltage ramp has a rate of change of 1kV/μsec, the separation of the capacitor plates is 100 μm and the radius of the capacitor is 55 mm, calculate B at $r = R = 55$ mm. Is this a large value? Can one compare typical values of B and E ?

Answers

Question 1.

EM radiation travelling through a dielectric travels at a speed of $c/\sqrt{\varepsilon_r}$ (see notes page 10-12). This means that the wavelength is also reduced by a factor of $\sqrt{\varepsilon_r}$. Thus for a given thickness d of dielectric there will be more wavelengths along the z-axis than in free space, so that the phase of the wave leaving the dielectric will lag the phase in free space. If we make this phase lag equal to an integral number of cycles, then the wave leaving the dielectric will be in phase with the wave that has travelled through the hole.

This assumes that there is no phase change during transmission through the interfaces- it can be seen from page 70 that for a lossless dielectric the transmission coefficient is real (not complex) and positive, so this is true.

The phase change due to propagation through a distance d (to be determined) of free space is:

$$\Delta \phi = 2\pi \frac{d}{\lambda} \qquad (1)$$

The phase change for propagation through an equal thickness of dielectric we will set to be equal to $\Delta \phi$ plus one cycle:

$$\Delta\phi + 2\pi = 2\pi \frac{d}{\lambda/\sqrt{\varepsilon_r}} \qquad (2)$$

Inserting (1) into (2) gives

$$2\pi \frac{d}{\lambda} + 2\pi = 2\pi \frac{d}{\lambda/\sqrt{\varepsilon_r}}$$
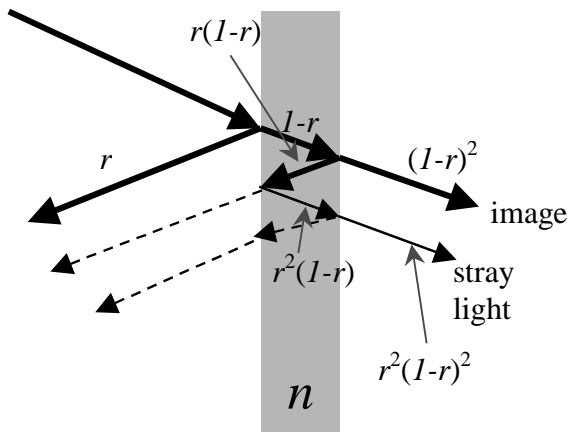
which we can solve to give

$$d + \lambda = d\sqrt{\varepsilon_r}$$

$$\Rightarrow d = \frac{\lambda}{1 - \sqrt{\varepsilon_r}}$$

Inserting values of $\lambda = c/f = 0.03$m and $\varepsilon_r = 2.7$ gives $d = 18.3$ mm.

Question 2

To estimate the farctional light powers transmitted and reflected from the lens, we can consider it as simply a glass plate:

The incident light will be partially reflected at both the air/glass and glass/air interface and some reflected light will bounce back and forth within the lens, with intensity diminishing at each reflection. We will consider only the power of the directly transmitted light ray and the first reflected ray that is transmitted towards the detector. The former of these forms the image and the latter is stray light, which causes ghost images and reduced contrast in the image.

The power reflectivity is equal to the square of the amplitude reflectivity (because power is proportional to E^2 – see page 25 for example) and is given by (see page 70):

$$r = \rho_E^2 = \left(\frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

The power reflectivity can be seen to be equal for both the air/glass and glass/air interfaces.

It can be seen from the above diagram and from the conservation of energy, that if a fraction r of the light intensity is reflected at an interface then $1-r$ is transmitted at the

air/glass interface and the same fraction 1-*r* is transmitted by the glass/air interface so that the light intensity at the detector will be reduced by a factor $(1-r)^2$ by the reflections.

The diagram also indicates the intensity of the multiply reflected beam that arrives at the detector and this can be seen to have an intensity that is reduced by a factor $r^2(1-r)^2$ since it has undergone two interface transmissions and two interface reflections.

For glass *n*=1.5, so the power reflectivity is

$$r = \rho_E^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 = \left(\frac{1 - 1.5}{1 + 1.5}\right)^2 = 0.04$$

and the fraction transmitted in the direct beam is $(1-0.04)^2=0.9216$.
The reflected beam has an intensity reduced by a factor of $0.04^2 \times (1-0.04)^2 = 0.00147$

For thermal imaging in the infrared, germanium lenses with a refractive index of 4 are often employed. The equivalent values are

$$r = \left(\frac{1 - 4}{1 + 4}\right)^2 = 0.36$$

directly transmitted intensity = $(1-0.36)^2=0.4096$
reflected intensity = $0.36^2 \times (1-0.36)^2 = 0.053$

Thus it can be seen that glass lenses transmit 92% of the incident light and the total power of the multiple reflections is only about 0.14% of that of the main image. Hence windows and spectacles give good performance without antireflection coating. However, for a germanium lens, only about 41% of the light is transmitted and the total power at the detector in the reflected light is about 13% of this. This becomes more significant if compound lenses with many surfaces are required. Thus germanium lenses are always antireflection coated. Antireflection coating is the subject of the next question.

Question 3

For an antireflection coating on a glass with a low *n*, we can approximate that the reflected intensity from the air/dielectric interface is equal to the reflected intensity from the dielectric/glass interface (that is we neglect the power lost by transmission through the air/glass interface). The double $\lambda/4$ delay through the dielectric ensures that the two componenets are π out of phase and cancel to give zero reflectivity at the design wavelength.

As an approximation, (which we will see works exactly) we set the two amplitude reflectivities to be exactly equal:
$$\frac{1 - n_d}{1 + n_d} = \frac{n_d - n_g}{n_d + n_g}$$
$$\Rightarrow (1 - n_d)(n_d + n_g) = (n_d - n_g)(1 + n_d)$$
$$\Rightarrow n_d - n_d^2 + n_g - n_d n_g = n_d + n_d^2 - n_g - n_d n_g$$
$$\Rightarrow n_d = \sqrt{n_g}$$

where n_d and n_g are the refractive indices of the dielectric and glass respectively. Thus the dielectric should have a refractive index equal to the geometric mean of the glass and the free space.

For the curious, it can be noted that the light actually reflects back and forth within the antireflection coating so that the resultant reflectivity is due to an infinite number of reflections that can be summed as a geometric progression. This is illustrated in the figure below.
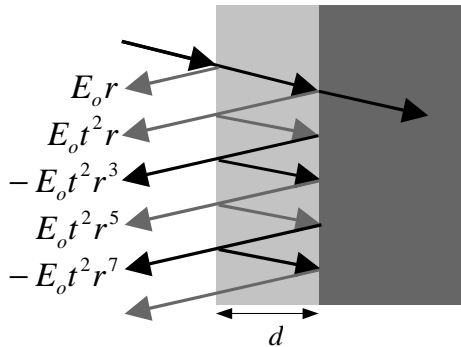
so that the reflected power is calculated by:

$$E_{or} = E_o r + E_o t^2 r e^{i2k\sqrt{n}d} - E_o t^2 r^3 e^{i4k\sqrt{n}d} + E_o t^2 r^5 e^{i6k\sqrt{n}d} - E_o t^2 r^7 e^{i8k\sqrt{n}d} + ...$$

$$= E_o r \left(1 + \left(1 - r^2\right)e^{i\delta}\left(1 - r^2 e^{i\delta} + r^4 e^{i2\delta} - r^6 e^{i3\delta} + ...\right)\right)$$

$$= E_o r \left(1 + \left(1 - r^2\right)e^{i\delta}\sum_{n=0}^{\infty}\left(-r^2 e^{i\delta}\right)\right) = E_o r \left(1 + \frac{\left(1 - r^2\right)e^{i\delta}}{1 + r^2 e^{i\delta}}\right)$$

$$= E_o r \frac{1 + e^{i\delta}}{1 + r^2 e^{i\delta}}$$

where we have used the following identities:

$r = r'$

$t = t'$

$t^2 = 1 - r^2$

$\delta = 2k\sqrt{n}d$

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad \left(\left|r\right| < 1\right)$$

Thus it can be seen that our assumption that zero reflectivity is obtained when the reflectivities at the air/dielectric and dielectric/glass interface are equal can be justified. However, this is presented only for the curious and is not expected as an answer.

The thickness of the dielctric is $\lambda/(4\sqrt{n_g})$. For =550 nm and n_g=1.5 this gives a thickness of 112.3 nm.

This antireflection coating will yield zero reflectivity at 550 nm, but at shorter wavelengths (that is blue light) and longer wavelengths (red light), light will be reflected resulting in a purple colour. This can be seen on antireflection coatings applied to camera lenses and glasses.

Question 4

Electromagnetic waves are generated by accelerating charges. In the case of light emitted by the sun, these charges are (generally) moving randomly with no preferred direction and hence light is emitted equally with all polarisation states with the net result that emitted

light is unpolarised. Charges accelerating within a dipole are constrained to the plane of the dipole and this results in radio waves that are plane polarised in the plane of the dipole.

Question 5

A linear polariser resolves the input light into the plane of polarisation transmitted by the polariser. That is, it passes that component of the input light that is polarised in the transmission plane of the polariser. Natural light has no preferred polarisation direction, and any free-space, plane electromagnetic wave has only two degrees of freedom (describing intensity), which we can call, for example, E_x and E_y, where the total intensity of the wave is given by $I = E_x^2 + E_y^2$. Because the light is unpolarised, light intensity is split equally between the x and y components and a linear polariser will transmit half of the input light.

The light transmitted through the linear polariser is now linearly polarised. The second polariser transmits the component of this light that is polarised in the direction of the transmission axis of the second polariser and similarly for the subsequent polarisers. Resolving components of one vector in the direction of another is simply obtained by taking the dot product of the vectors, which in normalised units, is equal to the cos of the angle between the vectors. Thus the total fraction of transmitted light is given by

$$\frac{I}{I_o} = \frac{1}{2}(\sin 30°)^3 = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^3 \approx 0.325$$

Question 6

The RMS power flow of EM radiation is given by $P = E^2/2\eta = \eta H^2/2$ (page 27).
Thus we get
$$E = \sqrt{2\eta P} = \sqrt{2.120\pi.1400} = 1027 \text{ V/m}$$
$$H = \sqrt{2P/\eta} = 2.72 \text{ A/m}$$
$$\Rightarrow B = \mu_o H = 3.48 \text{ T}$$

Question 7

At the dipole only the component of E pafrallel to the surface is zero (page 53,54) so the electric field lines are normal to the dipole. At distances much larger than the dipole length, the dipole appears as a point source as shown below. The intensity of the field lines falls with the inverse square of the displacement according to the Coulomb law.
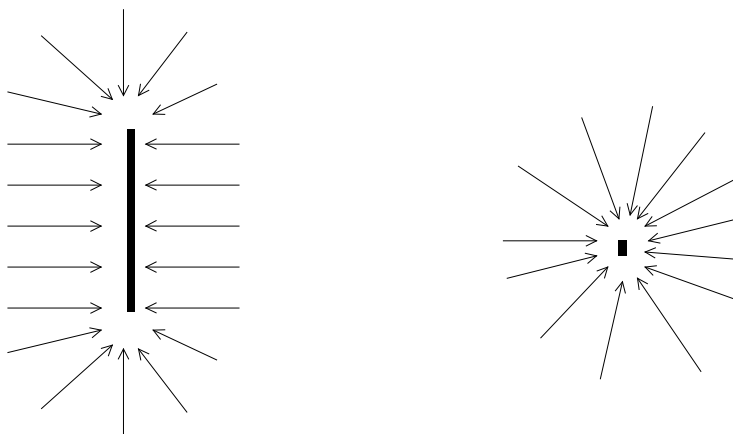
When a high frequency signal is applied to the dipole charges move along the dipole and the electric fields due to these charges move with the charges, distorting the electric field lines. These distortions, or wiggles propagate out at the speed of light. From Maxwells equations, the wiggles are displacement currents (d$\mathbf{E}$/dt) which set up transverse $\mathbf{H}$ fields (that loop right around the dipole) and the resulting d$\mathbf{H}$/dt set up transverse $\mathbf{E}$ fields and the net result is a propagating electromagnetic wave with transverse $\mathbf{E}$ and $\mathbf{H}$ fields – known as a tranverse electromagnetic wave (TEM). However, it is the second differential components $\partial^2 \mathbf{E}/\partial t^2$ and $\partial^2 \mathbf{H}/\partial t^2$ that mutually couple to produce the propagating TEM wave. The Coulomb **field** decays in strength with a $1/r^2$ law. Like any wave, the **power** in the propagating wave decays with an inverse square law, so the field (which is proportional to the square root of the power) decays as $1/r$; that is, much more slowly than the Coulomb rate of decay. This means that at large distances (that is many wavelengths) the Coulomb field is much weaker than the field due to the propagating wave and can be neglected. This is illustrated below. Close to the antenna, both near-field and propagating wave co-exist, but the near-field dominates, whereas at large distances, where the Coulomb field has decayed, the propagating wave dominates. It should be noted that an observer in the end-fire direction of the dipole does not see any wiggle in the field lines and hence no wave is propagated in that direction. See pages 4 & 5 and page 8 *et seq* of the lecture notes

Propagating electric fields

Wiggling Coulomb field
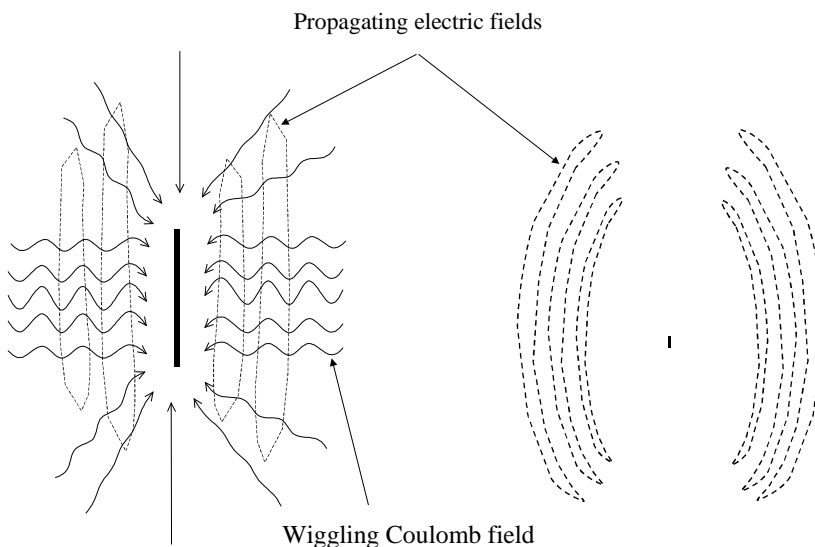
Question 8

Sketch page 21 of the notes for a diagram of the propagating E and H fields and for the description of the Poynting vector.

The pointing vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ gives the direction and power density of the propagating field.

We can write the power density as the power per unit area of the laser beam (if somewhat simplistically we assume it to be circular with uniform intrensity), thus:

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{E^2}{\eta} = \frac{P}{\pi r^2}$$

which can be rearranged to give

$$r = \sqrt{\frac{P\eta}{\pi E^2}}$$

Substituting in the maximum field strength and the power of the laser we obtain

$$r = \sqrt{\frac{1000.120\pi}{\pi.\left(3.10^6\right)^2}} = 115\mu\text{m}$$

Question 9

Ampers equation is

$$\oint \mathbf{H}.\mathbf{ds} = \iint_A \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \mathbf{dA} \ .$$

This states that, if we take a loop composed of an infinite number of infinitely small segments of length **ds**, measure the magnetic field vector **H** at the centre point of each segment, calculate the scalar product **H.ds** (that is $Hds\cos\vartheta$ where ϑ is the angle between the vector **H** and the vector **ds**) at each point and sum (integrate) the total, we obtain a value that is equal to the integral on the RHS. The integral on the RHS is the sum of two components, **J.dA** and $\partial \mathbf{D}/\partial t$.**dA** integrated over a surface **A** – this is the area of a surface for which the above mentioned loop is the perimeter. The surface is divided into an infinite number of infinitely small surfaces of area **dA.** For each of these surfaces we calculate the scalar product of the normal to the area **dA** and the current density **J** and the displacement current density $\partial \mathbf{D}/\partial t$ and sum (inegrate) all of these components to get the value of the RHS. Amperes law does not tell us how to choose the loop **S** or the surface **A**, it just simply states that if we choose a surface **A** with a perimeter **S** and then perform the integral on the LHS and perform the integral on the RHS they will be equal. However in cases with high degrees of symmetry we can conveniently choose a circlular perimeters **S** enclosing a disk **A** (or if appropriate, a rectangular perimeter encosing a rectangle) and the application of Amperes rule can very elegantly yield simple answers.

In this question we have circular symmetry so we will choose **A** to be a circle located between the plates of the capacitor and centred on the axis of the capacitor. There is no current **J** between the capacitor (i.e. **J**=0), but there is a changing electric field (a displacement current) due to the changing voltage applied to the plates. This is given by

$$E = \frac{V}{d}$$

$$\frac{\partial \mathbf{D}}{\partial t} = \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} = \frac{\varepsilon_o}{d}\frac{\partial V}{\partial t} = \frac{\varepsilon_o}{d}\alpha$$

and this is in the direction of the normal to the capacitor plates and our plane of integration. This assumes that electric field between the plates is uniform, which will be true for high conductivity plates. Each component **dA** of our area contributes a component

$$. = \partial \mathbf{D}/\partial t \, \mathbf{dA} \cos\vartheta = D \, da$$

$$\frac{\partial \mathbf{D}}{\partial t}.\mathbf{dA} = \frac{\partial D}{\partial t} dA \cos\vartheta = \frac{\varepsilon_o \alpha}{d} dA$$

to the RHS of the equation, so we obtain

$$RHS = \iint_A \frac{\varepsilon_o \alpha}{d} dA = \frac{\varepsilon_o \alpha}{d} \iint_A dA = \frac{\varepsilon_o \alpha}{d}\pi r^2 \ .$$

where the integral of the infinite number of infinitesimal components **dA** over **A** for a circle of radius r is simply πr^2.

To evaluate the LHS, we exploit the fact that radial symmetry means that **H** is constant around the perimeter of our circle and is always in the direction of the tangent to the circle so for all **ds**, **H.ds**=$H\,ds\,\cos\vartheta=Hds$ and

$$LHS = \oint \mathbf{H.ds} = H\oint ds = 2\pi rH = 2\pi r\frac{B}{\mu_o}$$

of course we could have chosen a totally different loop, for example a square loop, but we would not have been able to exploit symmetry so readily and so would not readily have obtained the answer in terms of r. Rigorous calculation, could however have yielded exactly the same answer (eventually!). Setting the LHS equal to the RHS yields

$$2\pi r\frac{B}{\mu_o} = \varepsilon_o \pi r^2 \frac{a}{d}$$

$$\Rightarrow B = \varepsilon_o \mu_o r\frac{a}{2d}$$

For values of $r>R$ the integral on the RHS is simply

$$RHS = \varepsilon_o \pi R^2 \frac{a}{d}$$

because the displacement current is zero outside of the confines of the capacitor (at least in this idealised example). Thus outside of the capacitor we obtain

$$2\pi r\frac{B}{\mu_o} = \varepsilon_o \pi r^2 \frac{a}{d}$$

$$\Rightarrow B = \varepsilon_o \mu_o \frac{R^2}{r}\frac{a}{2d}$$

Now the current flowing through the wires that charge the capacitor can be derived as follows:

$$Q = CV$$

$$I = \frac{dQ}{dt} = C\frac{dV}{dt} = C\alpha$$

$$= \frac{\varepsilon_o A}{d}\alpha = \frac{\varepsilon_o \pi R^2}{d}\alpha$$

where we have differentiated w.r.t. time, the charge stored on a capacitor so as to obtain an expression for the current in terms of the capacitance C and the rate of change of voltage α. We have then inserted the standard expression for the capacitance of a parallel plate capacitor, in terms of the plate area A. If we now choose an Amperian loop or radius r, centred on the wire (but larger than it so as to encompase all of the wire), the LHS of Amperes equation becomes simply $2\pi rH$ as before. To calculate the RHS, we note that the displacement current is zero (assuming a perfect conductor, so there is no electric field along the wires) and we must then integrate the current density **J** across the area of the wire and this simply gives the total current **I** (Again the direction of **J** is in the same direction to the normal of our surface **A**). Equating the LHS to the RHS simply gives

$$2\pi rH = 2\pi r\frac{B}{\mu_o} = I = \frac{\varepsilon_o \pi R^2}{d}\alpha$$

(This is the familiar $B = \mu_o I / 2\pi r$ for the magnetic field from an infinitely long current carrying wire). We can rearrange this to give

$$B = \frac{\varepsilon_o \mu_o R^2}{2rd} \alpha$$

which, as expected is exactly the same expression as that for the magnetic field due to the displacement current through the capacitor.

Thus, from a remote position, it is not possible to distinguish a displacement current from a current of electrons based on magnetic field measurement; that is, they are equivalent.

Inserting the values given yields

$$B = 8.85.10^{-12} \, 4\pi.10^{-7} \, 0.055 \frac{10^9}{2.100.10^{-6}} = 3.06\mu T$$

Although this is a very small number compared to probable values of *V* or $\partial E/\partial t$ one cannot compare one with the other since they have different units (similarly, one cannot say that 1 km is larger than a gram). However, it would be difficult to measure a field strength of this magnitude, so from that point of view it is small.

$$B = \frac{\varepsilon_o \mu_o R^2}{2rd} \alpha$$