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Standard Condition Number Distributions of Finite Wishart Matrices for Cognitive Radio Networks

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***Abstract*—The standard condition number (SCN) in random matrix theory (RMT) has widely been employed in cognitive radio networks (CRNs). The paper studies the SCN distributions of central and complex Wishart matrices with finite dimensions. The exact and closed-form expressions of SCN distributions are first proposed in sum of multiple polynomials. The proposed formulations provide an efficient way to determine probability density function and cumulative distribution function of the SCN in finite RMT. In particular, the specific SCN distributions of triple Wishart matrix are initially presented. The theoretical and precise sensing thresholds for cooperative spectrum sensing (CSS) systems in CRNs are calculated using the proposed SCN distributions. Numerical results indicate that the proposed theoretical SCN distributions match the empirical SCN distributions very well. Furthermore, the exact SCN-based CSS systems are able to obtain higher sensing performance than the asymptotic SCN-based schemes because precise sensing thresholds can be determined.**

***Index Terms*—Standard condition number, random matrix theory, wishart matrix, spectrum sensing, cognitive radio networks.**

I. Introduction

Random matrix theory (RMT), working as a sharp tool, has widely been used in many disciplines including signal processing, wireless communications, and even big data analysis [1]–[4]. Based on the dimensions of random matrices, RMT is approximately separated into two categories: infinite RMT (IRMT) and finite RMT (FRMT) [5], [6]. Some significant asymptotic distributions have been proposed for IRMT, such as Marchenko-Pasture law for general eigenvalues [7], Tracy-Widom law for extreme eigenvalues [8], and Tracy-Widom-Curtiss (TWC) law for standard condition number (SCN) [9]. However, for FRMT with small dimensions, the exact distributions are vital because the asymptotic results cannot precisely describe the features of finite random matrices [9]. The exact and analytic distributions including extreme eigenvalues, SCN, and Demmel condition number (DCN) are critical to some specific techniques including channel estimation in MIMO systems, spectrum sensing in CRNs, and signal processing for future 5G systems [9]–[11].

The asymptotic distributions of the SCN in IRMT can be achieved with the TWC law [9], in which the SCN is expressed as the ratio of two normalized extreme eigenvalues. The exact distributions of the SCN in FRMT were first discussed in [12] based on the joint probability density function (PDF) of all eigenvalues of Wishart matrices [13]. However, only the exact distributions of the SCN for the dual Wishart matrix with $K = 2$ were proposed in [12]. For the Wishart matrices with large dimensions (say, $K \geq 3$) in FRMT [6], the exact distributions of the SCN are still unknown to us. In this paper, the exact and closed-form SCN distributions of finite Wishart matrices with any $K \geq 3$ and $N \geq K$ are first proposed. The exact SCN distributions of triple Wishart matrix $(K = 3)$ are proposed to verify the proposed generalized distributions.

For practical use cases in CRNs, the RMT-based cooperative spectrum sensing (CSS) systems were discussed in [9], [14], [15]. The simulation results and analysis indicated that the *exact* SCN-based scheme outperforms the *asymptotic* counterpart since more precise theoretical thresholds can be calculated using exact SCN distributions [9]. In order to construct the SCN-based CSS systems with FRMT, the exact SCN distributions are critical to be determined.

II. Mathematical Preliminaries and System Model

The central and complex Wishart matrix $\mathbf{W} \sim \mathcal{W}_K(\mathbf{I}, N)$ with the dimension K, the degrees of freedom N $(N \geq K)$, and the identity covariance matrix $\mathbf{I}$ can be expressed as

$$\mathbf{W} \triangleq \mathbf{X} \cdot \mathbf{X}^{\dagger} \tag{1}$$

where $()^{\dagger}$ is the conjugate-transpose operation and each entry of the matrix $\mathbf{X}$ with the dimensions $K \times N$ follows a zero-mean complex Gaussian distribution. There are K non-negative and ordered eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_K$ of $\mathbf{W}$ and the SCN denoting by α can be defined as

$$\alpha = \frac{\lambda_K}{\lambda_1}. \tag{2}$$

For K ordered eigenvalues, the joint PDF can be expressed as [5], [13]

$$f_{\Gamma}(\lambda_1, \ldots, \lambda_K | K, N) = \prod_{k=1}^{K} \frac{\exp(-\lambda_k)\,\lambda_k^{N-K}}{(N-k)!(K-k)!} \prod_{k<n} (\lambda_n - \lambda_k)^2 \tag{3}$$

where k and n denote the indices of eigenvalues $\lambda_k \leq \lambda_n$ and $\mathbf{\Gamma} \triangleq \{\lambda_1, \ldots, \lambda_K\}$ denotes the set of K ordered eigenvalues.

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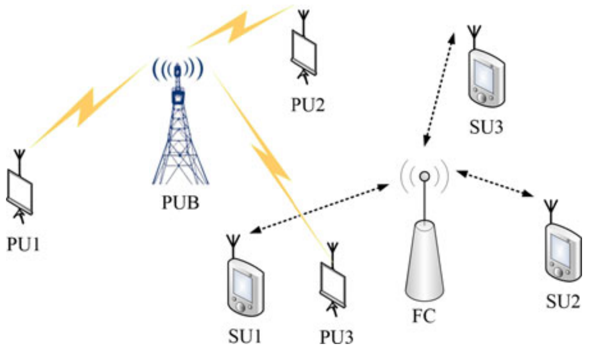

Fig. 1. Cooperative spectrum sensing model in CRNs.

Based on the joint PDF in (3), the exact distributions of the extreme eigenvalues [5, eqs. (26) and (30)] , the DCN [5, eq. (37)], and the scaled largest eigenvalue [5, eq. (42)] can be calculated. The SCN cumulative distribution function (CDF) in FRMT can be generated from multiple definite integrals of all K eigenvalues [12, eq. (37)], i.e.,

$$F_\alpha(x|K,N) = \Pr(\alpha \leq x) = \frac{1}{(K-1)!} \times \int_0^\infty \int_{\lambda_1}^{x\lambda_1} \cdots \int_{\lambda_1}^{x\lambda_1} f_{\mathbf{\Gamma}}(\lambda_1,\ldots,\lambda_K|K,N)\,\mathrm{d}\lambda_K\ldots\mathrm{d}\lambda_2\mathrm{d}\lambda_1 \tag{4}$$

where the integral interval for all variables (except λ_1) is located in the interval $[\lambda_1, x\lambda_1]$ and $f_{\mathbf{\Gamma}}(\lambda_1,\ldots,\lambda_K)$ is symmetric for all K arguments. For the dual Wishart matrix with $K=2$ and arbitrary integer $N \geq K$, [12] has provided the exact formulations of the SCN distributions.

In IRMT, exact SCN distributions cannot be achieved due to the large dimensions of Wishart matrix. The so-called TWC law has provided an asymptotic formulation for the normalized SCN [9], i.e.,

$$F_{\bar{\alpha}}(x|K,N) = \int_{\frac{\left(1+\sqrt{N/K}\right)^2}{u}}^{\infty} f_{\mathrm{TW}}(-y) \times F_{\mathrm{TW}}\left(\frac{x\left(\left(1+\sqrt{N/K}\right)^2 - uy\right) - \left(1-\sqrt{N/K}\right)^2}{v}\right)\mathrm{d}y \tag{5}$$

where $f_{\mathrm{TW}}(\cdot)$ and $F_{\mathrm{TW}}(\cdot)$ denote the PDF and CDF of the TW law, respectively. The normalized SCN $\bar{\alpha}$ is defined as

$$\bar{\alpha} \triangleq \frac{\lambda_K - \left(1+\sqrt{N/K}\right)^2}{\lambda_1 - \left(1-\sqrt{N/K}\right)^2} \cdot \sqrt[3]{\left(\frac{\sqrt{N}-\sqrt{K}}{\sqrt{N}+\sqrt{K}}\right)^4}. \tag{6}$$

In (5), u and v are expressed as

$$u \triangleq -\frac{\sqrt[3]{(\sqrt{N}-\sqrt{K})^4}}{K\sqrt[6]{KN}} \tag{7}$$

$$v \triangleq \frac{\sqrt[3]{(\sqrt{N}+\sqrt{K})^4}}{K\sqrt[6]{KN}}. \tag{8}$$

Based on the asymptotic CDF in (5), the distributions of the normalized SCN $\bar{\alpha}$ can be determined. Note that only the asymptotic CDF can be generated due to the unclosed formulation in (5).

The RMT-based CSS model is shown in Fig. 1, in which the CRNs including the fusion center (FC) and K secondary users (SUs) coexist with the primary user (PU) systems including the PU base-station (PUB) and some PUs. The SU working as distributed sensor can gather N PU signal samples for the FC. A sample matrix $\mathbf{X}$ with the dimension $K \times N$ is generated at the FC and the Wishart matrix $\mathbf{W}$ can be generated. The sensing performance of the CSS system can be evaluated by the hypothesis test, i.e.,

$$P_{\mathrm{d}} = \Pr\{T_\alpha \geq \phi|\mathcal{H}_1\} \tag{9}$$

where P_{d} denotes the probability of detection, the SCN-based sensor is defined as $T_\alpha \triangleq \lambda_K/\lambda_1$, the condition $\mathcal{H}_1$ denotes the presence of PU signals, and the threshold ϕ can be theoretically determined by

$$\phi = F_\alpha^{(-1)}\left[(1-P_{\mathrm{f}})\,|K,N\right]. \tag{10}$$

Here, $F_\alpha^{(-1)}(\cdot)$ is the CDF inverse function of the sensor T_α and P_{f} denotes the probability of false alarm. It should be noted that the exact theoretical CDF is critical to determine the precise thresholds, which are the key points to achieve higher sensing performance for CSS systems.

III. Exact SCN Distributions in FRMT

A. Exact SCN Distributions of Triple Wishart Matrix

The SCN distributions of triple central and complex Wishart matrix are presented as follows. Based on (4), the CDF of the SCN of $\mathbf{W} \sim \mathcal{W}_K(\mathbf{I}, N)$ with $K=3$ and arbitrary integer $N \geq K$ is given as

$$F_\alpha(x|3,N) = \frac{1}{2}\int_0^\infty \int_{\lambda_1}^{x\lambda_1} \int_{\lambda_1}^{x\lambda_1} f_{\mathbf{\Gamma}}(\lambda_1,\lambda_2,\lambda_3|3,N)\,\mathrm{d}\lambda_3\mathrm{d}\lambda_2\mathrm{d}\lambda_1 \tag{11}$$

where $\mathbf{\Gamma}$ denotes the eigenvalue set $\{\lambda_1,\lambda_2,\lambda_3\}$ and the joint PDF is expressed as

$$f_{\mathbf{\Gamma}}(\lambda_1,\lambda_2,\lambda_3|3,N) = (\lambda_1\lambda_2\lambda_3)^{N-3} \times \frac{\exp\left[-(\lambda_1+\lambda_2+\lambda_3)\right]\cdot\left[(\lambda_3-\lambda_2)(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)\right]^2}{\prod_{k=1}^{3}(3-k)!(N-k)!}. \tag{12}$$

The exact and closed-form CDF of the SCN can be expressed as

$$F_\alpha(x|3,N) = \xi(3,N)\sum_{r=1}^{R_3} L_3^r \sum_{\iota=1}^{\aleph[P(\mathbf{A}_3^r)]} \Psi_3^{\mathrm{F}}\left[P_\iota(\mathbf{A}_3^r), x\right] \tag{13}$$

where $\xi(3,N)$ is $1/[4(N-1)!(N-2)!(N-3)!]$, $R_3 = 5$ is the length of a coefficient set $\{1,2,-2,-2,-6\}$ and L_3^r is sequently taken from this set, the function $P_\iota(\cdot)$ denotes the l-th permutation of a set, and $\aleph[P(\mathbf{A}_3^r)]$ is the total number of the permutation of $\mathbf{A}_3^r$. The parameter sets are given as $\mathbf{A}_3^1 = \{N-3, N-1, N+1\}$, $\mathbf{A}_3^2 = \{N-2, N-1, N\}$, $\mathbf{A}_3^3 = \{N-2, N-2, N+1\}$, $\mathbf{A}_3^4 = \{N-3, N, N\}$, and $\mathbf{A}_3^5 = \{N-1, N-1, N-1\}$. Let a, b, and c denote a permutation of the parameter set, i.e.,

$$\{a,b,c\} = P_\iota(\mathbf{A}_3^r). \tag{14}$$

Hence, $\Psi_3^{\mathrm{F}}\left[P_\iota(\mathbf{A}_3^r), x\right]$ is given by

$$\Psi_3^{\mathrm{F}}[a,b,c,x] = \sum_{i=0}^{a}\sum_{j=0}^{b} i!j!\binom{a}{i}\binom{b}{j} \times (a+b+c-i-j)!\,\Theta_{\mathrm{F}}(x|a,b,c;i,j;3,N) \tag{15}$$

where

$$\Theta_{\mathrm{F}}(x|a,b,c;i,j;3,N) = \frac{x^{a+b-i-j}}{(2x+1)^{a+b+c-i-j+1}} + \frac{1}{3^{a+b+c+1-i-j}} - \frac{x^{a-i}+x^{b-j}}{(x+2)^{a+b+c-i-j+1}}. \quad (16)$$

Proof: See the Appendix. ∎

The exact and closed-form PDF of the SCN $f_\alpha(x|3,N)$ can be generated by

$$f_\alpha(x|3,N) = \frac{\mathrm{d}\left[F_\alpha(x|3,N)\right]}{\mathrm{d}x} = \xi(3,N)\sum_{r=1}^{R_3} L_3^r \sum_{\iota=1}^{\aleph\left[P\left(\mathbf{A}_3^r\right)\right]} \Psi_3^{\mathrm{f}}\left[P_\iota\left(\mathbf{A}_3^r\right),x\right] \quad (17)$$

where $\Psi_3^{\mathrm{f}}\left[P_\iota\left(\mathbf{A}_3^r\right),x\right]$ is defined as

$$\Psi_3^{\mathrm{f}}(a,b,c,x) = \sum_{i=0}^{a}\sum_{j=0}^{b} i!j!\binom{a}{i}\binom{b}{j} \times (a+b+c-i-j)!\Theta_{\mathrm{f}}(x|a,b,c;i,j;3,N). \quad (18)$$

Here, $\Theta_{\mathrm{f}}(x|a,b,c;i,j;3,N)$ is expressed as

$$\Theta_{\mathrm{f}}(x|a,b,c;i,j;3,N) = \frac{x^{\mu-c-1}[(\mu-c)(2x+1)-2(\mu+1)x]}{(2x+1)^{\mu+2}} + \frac{(\mu+1)(x^{a-i}+x^{b-j})-(x+2)[(a-i)x^{a-i-1}+(b-j)x^{b-j-1}]}{(x+2)^{\mu+2}} \quad (19)$$

where $\mu = a+b+c-i-j$.

B. Generalized SCN Distributions of Finite Wishart Matrix

For finite Wishart matrix with the dimension $K > 3$, the exact SCN distributions can still be calculated using the above approach. The final expression is expressed as the multiple homogeneous polynomials. Based on the joint PDF in (3) and the corresponding CDF in (4), the closed-form formulations of the SCN distributions can be calculated. The key part of the integrand is $\left[\prod_{1\le k<n\le K}^{K}(\lambda_n-\lambda_k)\right]^2$, in which there are in total $3^{\frac{K(K-1)}{2}}$ terms each with the degree of K.

The closed-form SCN CDF of the Wishart matrix with a finite K and an arbitrary N is expressed as

$$F_\alpha(x|K,N) = \xi(K,N)\sum_{r=1}^{R_K} L_K^r \sum_{\iota=1}^{\aleph\left[P\left(\mathbf{A}_K^r\right)\right]} \Psi_K^{\mathrm{F}}\left[P_\iota\left(\mathbf{A}_K^r\right),x\right] \quad (20)$$

where $\xi(K,N)$ is $1/\left[(K-1)!\prod_{k=1}^{K}(K-k)!(N-k)!\right]$, R_K is the number of the categories of $\mathbf{A}_K^r$, $r=1,\ldots,R_K$ denotes the indices, the coefficient L_K^r is determined by the category of $\mathbf{A}_K^r$, $\aleph\left[P\left(\mathbf{A}_K^r\right)\right]$ is the total number of the permutation of $\mathbf{A}_K^r$, and $P_\iota\left(\mathbf{A}_K^r\right)$ denotes the l-th permutation of the K-length set of $\{a_1,\ldots,a_K\}$. Here, the function $\Psi_K^{\mathrm{F}}\left[P_\iota\left(\mathbf{A}_K^r\right),x\right]$ is defined as

$$\Psi_K^{\mathrm{F}}\left[P_\iota\left(\mathbf{A}_K^r\right),x\right] = \Psi_K^{\mathrm{F}}\left[a_1,\ldots,a_K,x\right] = \sum_{i_1=0}^{a_1}\sum_{i_2=0}^{a_2}\cdots\sum_{i_{K-1}=0}^{a_{K-1}} i_1!i_2!\ldots i_{K-1}!\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_{K-1}}{i_{K-1}} \left(\sum\mathbf{B}+a_K\right)!\Theta_{\mathrm{F}}(x|a_1,\ldots,a_K;i_1,\ldots,i_{K-1};K,N) \quad (21)$$

where $\mathbf{B}$ denotes a $K-1$ length set $\{a_1-i_1, a_2-i_2\ldots,a_{K-1}-i_{K-1}\}$ and $\sum\mathbf{B}$ is the sum of all entries. The function $\Theta_{\mathrm{F}}(x|a_1,\ldots,a_K;i_1,\ldots,i_{K-1};K,N)$ is given as

$$\Theta_{\mathrm{F}}(x|a_1,\ldots,a_K;i_1,\ldots,i_{K-1};K,N) = \sum_{j=1}^{K}\sum_{u=1}^{\aleph\left[C_{K-1}^{j-1}(\mathbf{B})\right]} \frac{(-1)^{(K-j)}x^{\left[\sum\mathbf{B}-\sum\mathbf{D}_j^u\right]}}{[(K-j)x+j]^{\sum\mathbf{B}+a_K+1}} \quad (22)$$

where $\aleph\left[C_{K-1}^{j-1}(\mathbf{B})\right] = \binom{K-1}{j-1}$ denotes the number of the combinations of $j-1$ entries from $\mathbf{B}$, $\mathbf{D}_j^u$ denotes the u-th subset of $\mathbf{B}$. When $j=1$, $\aleph\left[C_{K-1}^{0}(\mathbf{B})\right]=1$ and $\mathbf{D}_1^1=\emptyset$. If $j=K$, $\aleph\left[C_{K-1}^{K-1}(\mathbf{B})\right]=1$ and $\mathbf{D}_{K-1}^1=\mathbf{B}$.

Proof: The CDF in (20) can be proved by the method of mathematical induction. ∎

The corresponding PDF can be achieved by

$$f_\alpha(x|K,N) = \frac{\mathrm{d}F_\alpha(x|K,N)}{\mathrm{d}x} = \xi(K,N)\sum_{r=1}^{R_K} L_K^r \sum_{\iota=1}^{\aleph\left[P\left(\mathbf{A}_K^r\right)\right]} \Psi_K^{\mathrm{f}}\left[P_\iota\left(\mathbf{A}_K^r\right),x\right] \quad (23)$$

where $\Psi_K^{\mathrm{f}}\left[P_\iota\left(\mathbf{A}_K^r\right),x\right]$ is defined as

$$\Psi_K^{\mathrm{f}}\left[P_\iota\left(\mathbf{A}_K^r\right),x\right] = \Psi_K^{\mathrm{f}}\left[a_1,\ldots,a_K,x\right] = \sum_{i_1=0}^{a_1}\sum_{i_2=0}^{a_2}\cdots\sum_{i_{K-1}=0}^{a_{K-1}} i_1!i_2!\ldots i_{K-1}!\binom{a_1}{i_1}\binom{a_2}{i_2}\cdots\binom{a_{K-1}}{i_{K-1}} \left(\sum\mathbf{B}+a_K\right)!\Theta_{\mathrm{f}}(x|a_1,\ldots,a_K;i_1,\ldots,i_{K-1};K,N). \quad (24)$$

The function $\Theta_{\mathrm{f}}(x|a_1,\ldots,a_K;i_1,\ldots,i_{K-1};K,N)$ is given as

$$\Theta_{\mathrm{f}}(x|a_1,\ldots,a_K;i_1,\ldots,i_{K-1};K,N) = \sum_{j=1}^{K}\sum_{u=1}^{\aleph\left[C_{K-1}^{j-1}(\mathbf{B})\right]} \frac{(-1)^{(K-j)}x^{\left[\sum\mathbf{B}-\sum\mathbf{D}_j^u-1\right]}}{[(K-j)x+j]^{\sum\mathbf{B}+a_K+1}} \left[\sum\mathbf{B}-\sum\mathbf{D}_j^u-\frac{x(K-j)\left(\sum\mathbf{B}+a_K+1\right)}{x(K-j)+j}\right]. \quad (25)$$

Note that the CDF and PDF for $K=3$ and an arbitrary N in (13) and (17) are just specific expressions with minor modifications of (20) and (23), respectively.

IV. Numerical Results

The theoretical SCN distributions in FRMT are verified with the empirical SCN distributions. The CDFs and PDFs of the SCN of the Wishart matrices with $K=3$ and various integers $N=6,8,10$ are indicated in Figs. 2 and 3, respectively. In Fig. 2, the theoretical distributions based on the exact and closed-form formulation $F_\alpha(x|3,N)$ in (13) can completely match the empirical distributions. It can be noted that with a larger N, the SCN takes smaller values. In Fig. 3, the theoretical PDFs $f_\alpha(x|3,N)$ generated by (17) fit exactly the empirical results. With the exact CDFs and PDFs of the SCN, the characteristics of the SCN of finite Wishart matrices can be evaluated.

Based on the model in Fig. 1 and (9), the simulations are performed. The sensing performance is evaluated in Fig. 4, in which the SCN-based scheme is compared with the TWC-based scheme. The SNR was set to −10:2:10 dB and P_{f} was set to 0.05. The number of SUs

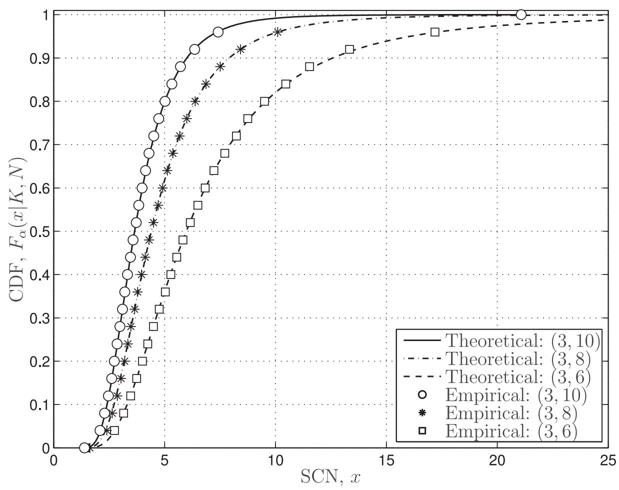

Fig. 2. Exact CDFs of SCN for Wishart matrices with various dimensions (K, N).

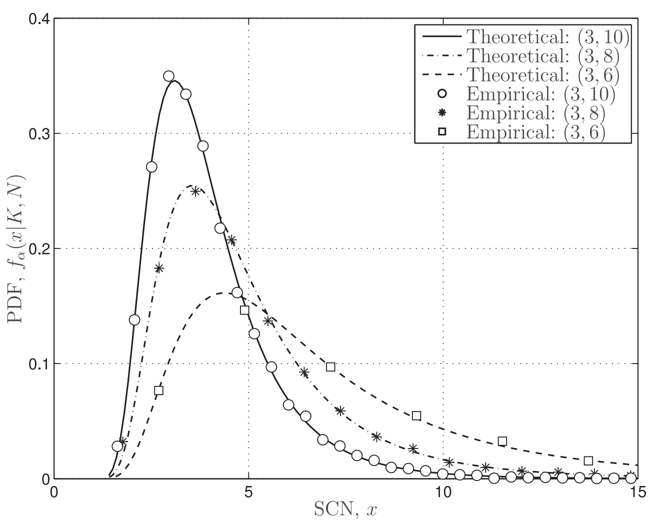

Fig. 3. Exact PDFs of SCN for Wishart matrices with various dimensions (K, N).

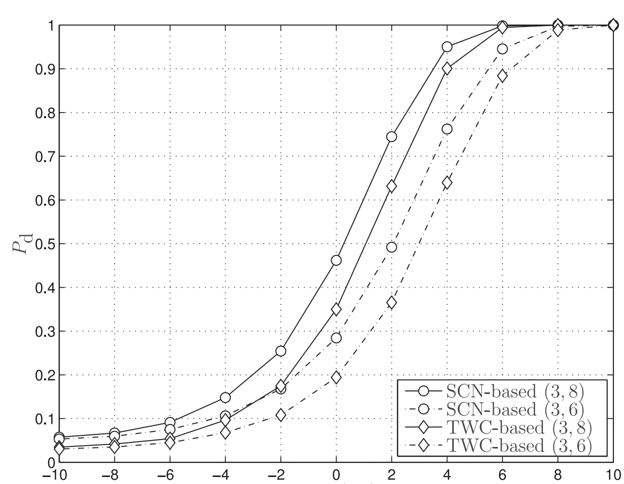

Fig. 4. Cooperative spectrum sensing performance with SCN vs. TWC for various dimensions (K, N).

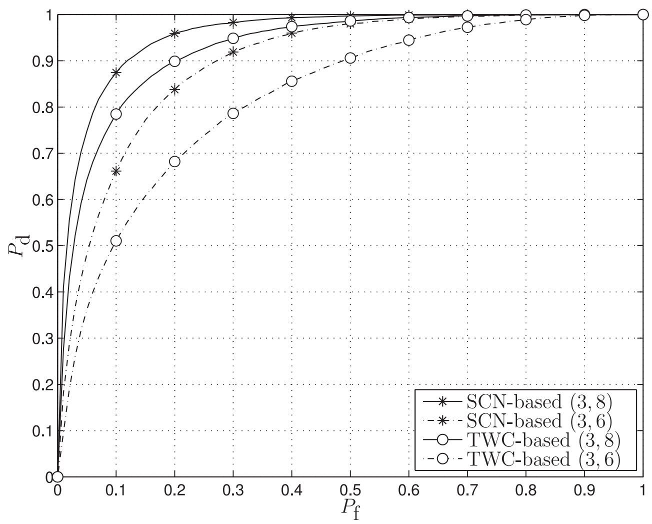

Fig. 5. Receiver operating characteristic performance with SCN vs. TWC for various dimensions (K, N).

was set to $K = 3$ and the number of PU signal samples was set to $N = 6$ or 8, i.e., each SU sent 6 or 8 PU signal samples to the FC and a 3×3 sample matrix was generated. It can be found that the SCN-based scheme outperforms the TWC-based scheme. For example, there is about 2 dB performance gain when $K = 3, N = 8$, and the SNR is 0 dB. The receiver operating characteristic performance is shown in Fig. 5, in which the SNR of PU signal is 2 dB. The SCN-based scheme outperforms the TWC-based scheme for all P_f. The reason is that the SCN-based scheme utilizes more precise thresholds than the TWC-based scheme. The unclosed formulation in (5) cannot generate the theoretical thresholds.

V. Conclusion

The exact and closed-form formulations of the SCN distributions of the central and complex Wishart matrices with finite dimensions have been deduced in this paper. The proposed exact SCN distributions and the asymptotic SCN distributions provided by the TWC law can completely determine the characteristics of the SCN of the Wishart matrices with any dimensions.

Appendix
Proof of (13)

The joint PDF in (12) can be rewritten as

$$f_\Gamma(\lambda_1, \lambda_2, \lambda_3|3, N) = \xi(3, N)(\lambda_1\lambda_2\lambda_3)^{N-3} \times \exp(-\lambda_1 - \lambda_2 - \lambda_3)\, G(\lambda_1, \lambda_2, \lambda_3) \quad (26)$$

where the function $\xi(3, N)$ is $\frac{1}{2(N-1)!(N-2)!(N-3)!}$ and $G(\lambda_1, \lambda_2, \lambda_3)$ can be expressed as a homogeneous polynomial

$$\begin{aligned} G(\lambda_1, \lambda_2, \lambda_3) = {} & [(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)]^2 \\ = {} & \lambda_3^4\lambda_2^2 + \lambda_1^4\lambda_2^2 + \lambda_1^2\lambda_2^4 + \lambda_2^4\lambda_3^2 + \lambda_1^4\lambda_3^2 + \lambda_1^2\lambda_3^4 \\ & + 2(\lambda_1^3\lambda_2^2\lambda_3 + \lambda_1^2\lambda_2^3\lambda_3 + \lambda_1\lambda_2^3\lambda_3^2 + \lambda_1^3\lambda_2\lambda_3^2 \\ & + \lambda_1^2\lambda_2\lambda_3^3 + \lambda_1\lambda_2^2\lambda_3^3) - 2(\lambda_1\lambda_2^4\lambda_3 + \lambda_1^4\lambda_2\lambda_3 \\ & + \lambda_1\lambda_2\lambda_3^4 + \lambda_1^3\lambda_2^3 + \lambda_2^3\lambda_3^3 + \lambda_1^3\lambda_3^3) - 6\lambda_1^2\lambda_2^2\lambda_3^2. \end{aligned} \quad (27)$$

Here, there are in total 19 terms each with the degree of 6. Based on (11), each term of the polynomial can be integrated. Let us take the

$$\begin{aligned}\Omega_1(x) &= \int_0^{\infty}\int_{\lambda_1}^{x\lambda_1}\int_{\lambda_1}^{x\lambda_1}(\lambda_1\lambda_2\lambda_3)^{N-3}\exp(-\lambda_1-\lambda_2-\lambda_3)\lambda_3^4\lambda_2^2\mathrm{d}\lambda_3\mathrm{d}\lambda_2\mathrm{d}\lambda_1\\ &= \int_0^{\infty}\exp(-\lambda_1)\lambda_1^{N-3}\left[\int_{\lambda_1}^{x\lambda_1}\lambda_2^{N-1}\exp(-\lambda_2)\mathrm{d}\lambda_2\int_{\lambda_1}^{x\lambda_1}\lambda_3^{N+1}\exp(-\lambda_3)\mathrm{d}\lambda_3\right]\mathrm{d}\lambda_1. \end{aligned} \tag{28}$$

$$\begin{aligned}\Delta(x) = &\left[-\exp(-x\lambda_1)\left(\sum_{k=0}^{N-1}k!\binom{N-1}{k}(x\lambda_1)^{N-1-k}\right)+\exp(-\lambda_1)\left(\sum_{k=0}^{N-1}k!\binom{N-1}{k}\lambda_1^{N-1-k}\right)\right]\\ &\times\left[-\exp(-x\lambda_1)\left(\sum_{m=0}^{N+1}m!\binom{N+1}{m}(x\lambda_1)^{N+1-m}\right)+\exp(-\lambda_1)\left(\sum_{m=0}^{N+1}m!\binom{N+1}{m}\lambda_1^{N+1-m}\right)\right]. \end{aligned} \tag{31}$$

first term $\lambda_3^4\lambda_2^2$ in (27) as an example to illustrate the integral process, which is shown in (28) at the top of this page. Let $\Delta(x)$ denote the integrand, i.e.,

$$\Delta(x) = \int_{\lambda_1}^{x\lambda_1}\lambda_2^{N-1}\exp(-\lambda_2)\mathrm{d}\lambda_2\int_{\lambda_1}^{x\lambda_1}\lambda_3^{N+1}\exp(-\lambda_3)\mathrm{d}\lambda_3. \tag{29}$$

According to [16, eq. (2.321.2)]

$$\int x^n\exp(ax)\,dx = \exp(ax)\left(\sum_{k=0}^{n}\frac{(-1)^k k!\binom{n}{k}}{a^{k+1}}x^{n-k}\right) \tag{30}$$

$\Delta(x)$ in (29) can be rewritten as (31) shown at the top of this page.

After some algebraic calculations and simplifications, $\Omega_1(x)$ can be calculated as

$$\begin{aligned}\Omega_1(x) &= \int_0^{\infty}\exp(-\lambda_1)\,\lambda_1^{N-3}\Delta(x)\,\mathrm{d}\lambda_1\\ &= \sum_{i=0}^{N-1}\sum_{j=0}^{N+1} i!j!\binom{N-1}{i}\binom{N+1}{j}(3N-3-i-j)!\\ &\quad\Xi(a,b,c,x,i,j) \end{aligned} \tag{32}$$

where $\Xi(a,b,c,x,i,j)$ can be formulated as

$$\begin{aligned}\Xi(a,b,c,x,i,j) &= \frac{x^{2N-i-j}}{(2x+1)^{3N-2-i-j}}\\ &\quad+\frac{1}{3^{3N-2-i-j}}-\frac{x^{N-1-i}+x^{N+1-j}}{(x+2)^{3N-2-i-j}}. \end{aligned} \tag{33}$$

Based on (32) and (33), $\Psi_3^F[a,b,c,x]$ in (15) and $\Theta_{\mathrm{F}}(x|a,b,c;i,j;3,N)$ in (16) can be achieved. For the first term $\lambda_3^4\lambda_2^2$, $a=N-1$, $b=N+1$, and $c=N-3$. For other 18 terms, the parameters a, b, and c can be generated with the permutation operation of 5 parameter sets $\mathbf{A}_3^r, r=1,\ldots,5$. Therefore, the closed-form CDF in (13) can be achieved.

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