SHOCK STRUCTURE DESCRIPTION USING RE-CASTED NAVIER-STOKES EQUATIONS

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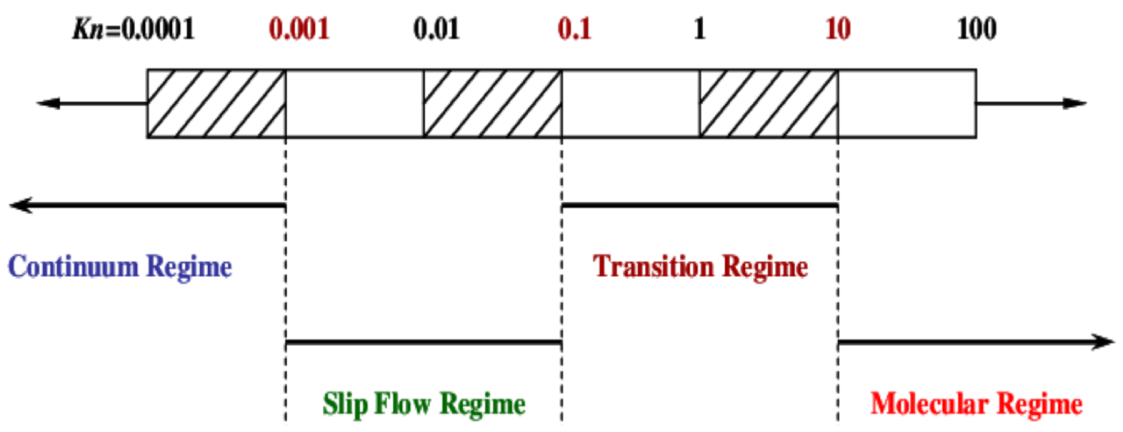
Introduction



- **Classical NS equations are known to be inadequate in describing some compressible flow configurations.**
- **One of the best-known example of NS failure, is the shock structure description.**
- **Reason for failure: typical values of Kn for the flow with-in shock layer falls between 0.2 and 0.3.**

Navier-Stokes Equations

■ **Continuity equation:**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho U] = 0. \qquad (1)$$

■ **Momentum balance equation:**

$$\frac{\partial \rho U}{\partial t} + \nabla \cdot [\rho U \otimes U] + \nabla \cdot [p \boldsymbol{I} + \boldsymbol{\Pi}^{(NS)}] = 0. \qquad (2)$$

■ **Energy balance equation:**

$$\frac{\partial}{\partial t}[\frac{1}{2}\rho U^2 + \rho \mathbf{e}_{in}] + \nabla \cdot [\frac{1}{2}\rho U^2 U + \rho \mathbf{e}_{in} U] + \nabla \cdot [(p\boldsymbol{I} + \boldsymbol{\Pi}^{(NS)}) \cdot U] + \nabla \cdot \boldsymbol{q}^{(NS)} = 0. \qquad (3)$$

$$\boldsymbol{\Pi}^{(NS)} = -2\mu \left[\frac{1}{2}(\nabla U + \widetilde{\nabla U}) - \frac{1}{3}\boldsymbol{I}\,(\nabla \cdot U)\right],$$

$$\boldsymbol{q}^{(NS)} = -\kappa \nabla T.$$

Re-casted Navier-Stokes

Mass-diffusion Navier-Stokes: $U \rightarrow U_v - \kappa_m \nabla \ln \rho$

■ **Re-casted continuity equation:**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho U_v] = \kappa_m \Delta \rho. \qquad (4)$$

■ **Re-casted momentum balance equation:**

$$\frac{\partial \rho U_v}{\partial t} + \nabla \cdot [\rho U_v \otimes U_v] + \nabla \cdot \left[p\boldsymbol{I} + \boldsymbol{\Pi}_v^{(RNS)}\right] - \kappa_m^2 \nabla \Delta \rho + \kappa_m \nabla[\nabla \cdot (\rho U_v)] = 0, \qquad (5)$$

$$\boldsymbol{\Pi}_v^{(RNS)} = \boldsymbol{\Pi}_v + \frac{\kappa_m^2}{\rho}\nabla\rho \otimes \nabla\rho - \kappa_m U_v \otimes \nabla\rho - \kappa_m \nabla\rho \otimes U_v,$$

$$\boldsymbol{\Pi}_v = -2\mu\overline{\overset{\circ}{\nabla U_v}} + 2\mu\kappa_m \widetilde{\boldsymbol{D}} \ln\rho - \frac{2\mu}{3}\kappa_m \Delta \ln \rho\, \boldsymbol{I}.$$

■ **Re-casted energy balance equation:**

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho U_v^2 + \rho \mathbf{e}_{in}\right] + \nabla \cdot \left[\frac{1}{2}\rho U_v^2 U_v + \rho \mathbf{e}_{in} U_v\right] + \nabla \cdot \left[(p\boldsymbol{I} + \boldsymbol{\Pi}_v) \cdot U_v - \kappa_m \boldsymbol{\Pi}_v \cdot \nabla \ln \rho\right] + \nabla \cdot \left[\boldsymbol{q}_v^{(RNS)}\right] + \nabla \cdot \left[\kappa_m \mathcal{N}_{v_1} + \kappa_m^2 \mathcal{N}_{v_2} + \kappa_m^3 \mathcal{N}_{v_3}\right] + \kappa_m \mathcal{N}_{v_4} + \kappa_m^2 \mathcal{N}_{v_5} + \kappa_m^3 \mathcal{N}_{v_6} = 0, \qquad (6)$$

$$\boldsymbol{q}_v^{(RNS)} = \boldsymbol{q}^{(NS)} - \kappa_m \rho \mathbf{e}_{in} \nabla \ln\rho - \kappa_m p \boldsymbol{I} \cdot \nabla \ln \rho,$$

$$\mathcal{N}_{v_1} = -(U_v \cdot \nabla\rho) U_v - \frac{1}{2} U_v^2 \nabla\rho,$$

$$\mathcal{N}_{v_2} = (U_v \cdot \nabla\rho)\nabla \ln \rho + \frac{1}{2\rho}|\nabla\rho|^2 U_v,$$

$$\mathcal{N}_{v_3} = -\frac{1}{2\rho}|\nabla\rho|^2 \nabla \ln \rho,$$

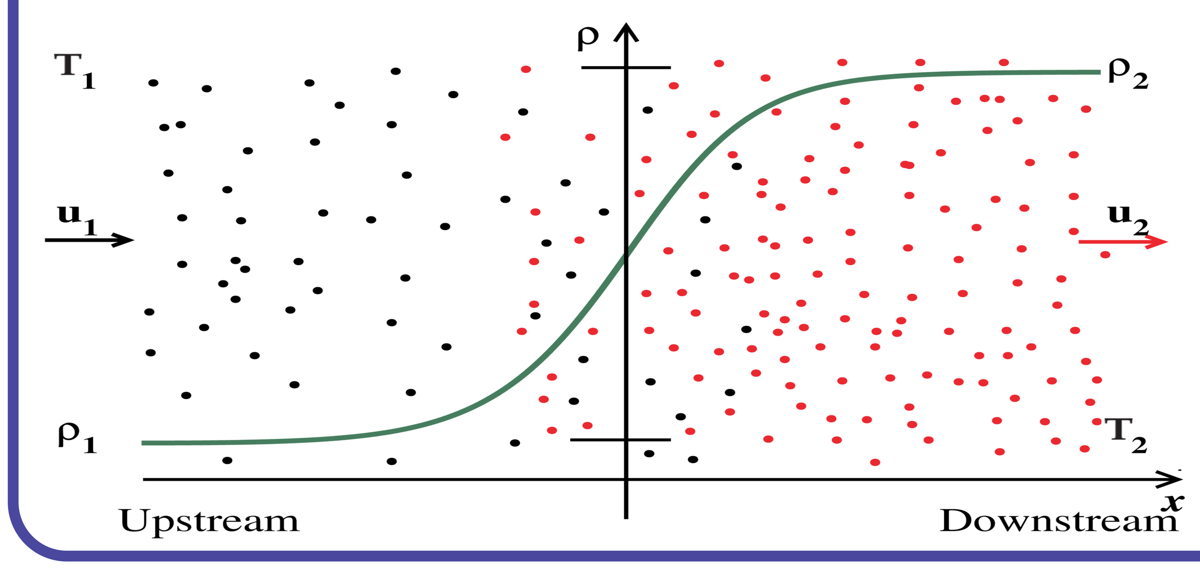
$$\mathcal{N}_{v_4} = \nabla \cdot \left[\rho U_v \otimes U_v + p\boldsymbol{I} + \boldsymbol{\Pi}_v^{(RNS)}\right] \cdot \nabla \ln\rho - U_v \cdot \left[\nabla \ln\rho \nabla \cdot (\rho U_v) - \nabla(\nabla \cdot (\rho U_v))\right],$$

$$\mathcal{N}_{v_5} = (U_v \cdot \Delta\rho \nabla \ln \rho) - (U_v \cdot \nabla\Delta\rho) + \frac{1}{2}\frac{|\nabla\rho|^2}{\rho^2}\nabla \cdot (\rho U_v),$$

$$\mathcal{N}_{v_6} = -\frac{1}{2\rho^2}|\nabla\rho|^2 \Delta\rho.$$

Stationary Shock Wave Problem

■ The shock wave structure problem is one of the simplest example for highly non-equilibrium flow phenomena.
■ Shock wave is an interface of finite thickness between two different equilibrium states of a gas.



■ Reference Variables:

$$\bar{\rho} = \frac{c_1^2}{p_1}\rho = \frac{\gamma}{\rho_1}\rho, \quad \bar{u}_v = \frac{u_v}{c_1}, \quad \bar{T} = \frac{\boldsymbol{R}}{c_1^2}T, \quad \bar{p} = \frac{p}{p_1},$$

$$\bar{x} = \frac{x}{\lambda_1}, \quad \bar{\mu} = \frac{\mu}{\mu_1}, \quad \bar{\kappa} = \frac{\gamma}{(\gamma - 1)\,\mathrm{Pr}}\bar{\mu}, \quad \bar{\kappa}_m = -\kappa_{m_0}\frac{\bar{\mu}}{\bar{\rho}}.$$

where $\bar{p} = \bar{\rho}\bar{T}$, λ_1 is the upstream mean free path, $c_1 = \sqrt{\gamma \boldsymbol{R} T_1}$ being the adiabatic sound speed.
■ The viscosity-temperature relation: $\mu \propto T^s$
For a monatomic Argon gas here we take $s = 0.75$
■ The upstream Mach number Ma_1 is defined as $\mathrm{Ma}_1 = |u_{v_1}|/c_1$.

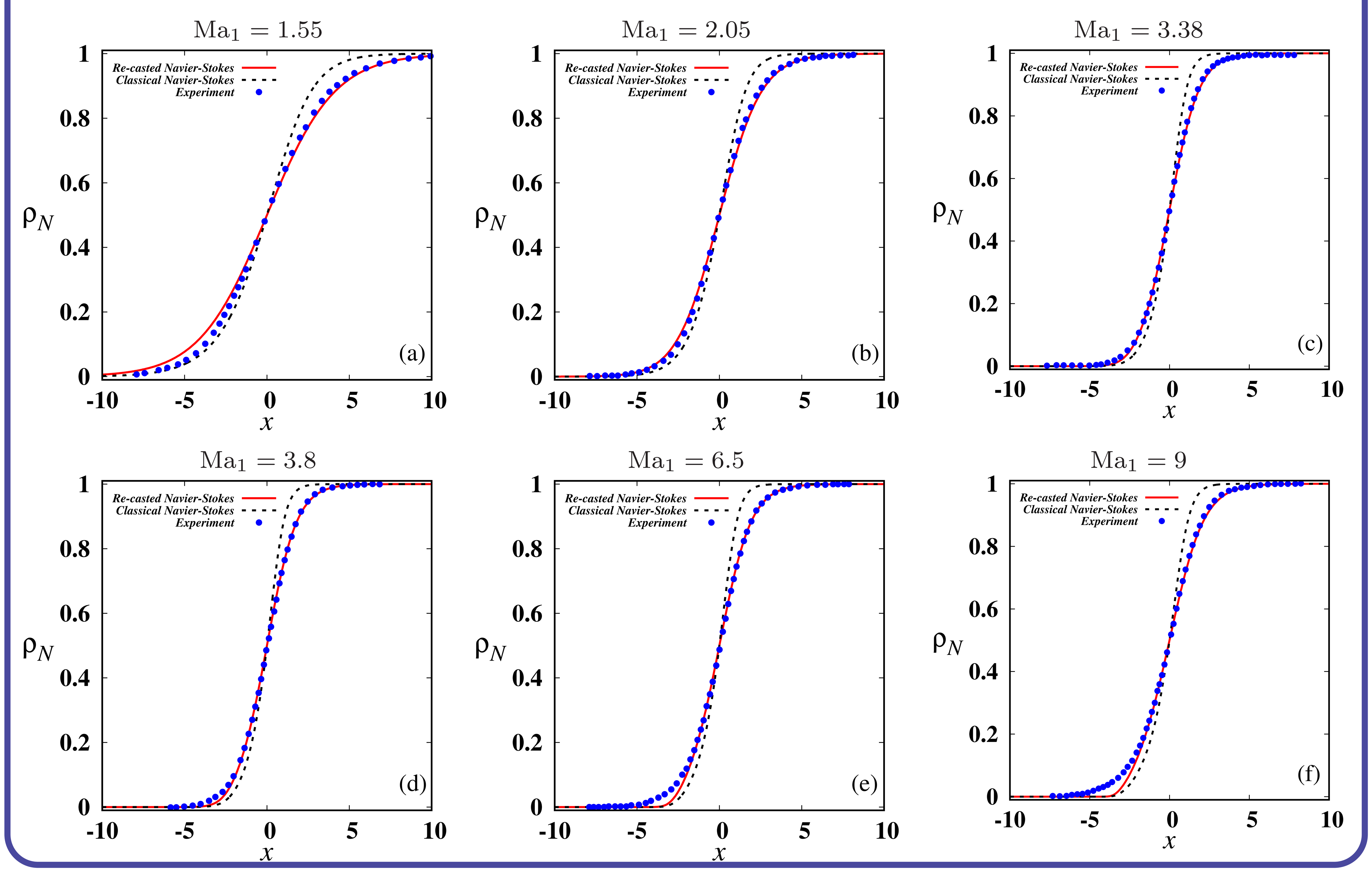
Reduced Re-casted NS Equations

$$\left(1 + \frac{\bar{\kappa}_m}{\lambda_0 \mathrm{Ma}_1}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}\bar{x}}\right)\bar{u}_v + \frac{1}{\bar{m}_0}\bar{\rho}\bar{T} + \frac{1}{\lambda_0 \mathrm{Ma}_1}\bar{\Pi}_v^{(RNS)} - \bar{p}_0 = 0, \qquad (7)$$

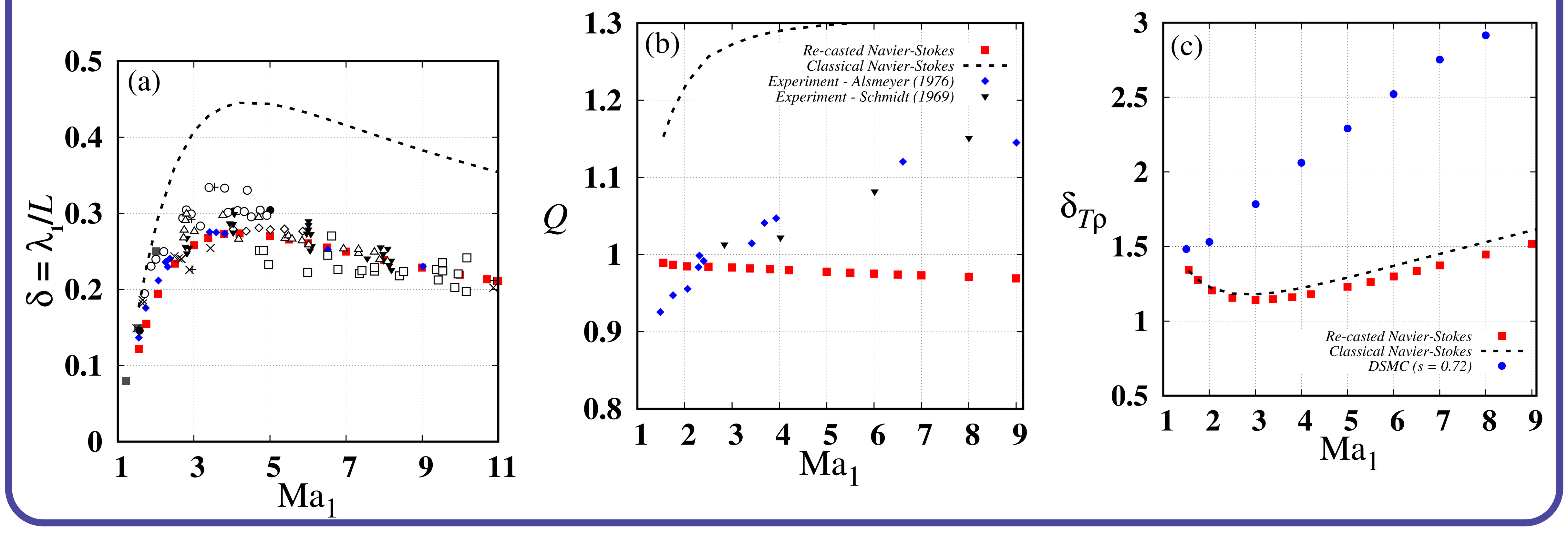
$$\left(1 + \frac{\bar{\kappa}_m}{\lambda_0 \mathrm{Ma}_1}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}\bar{x}}\right)\left(\frac{\gamma}{(\gamma-1)}\bar{T} + \frac{1}{2}\bar{u}_v^2\right) - \frac{\gamma^2}{\lambda_0^3 \mathrm{Ma}_1}\frac{\bar{\kappa}_m^3}{2\bar{\rho}^2}\left(\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}\bar{x}}\right)^3 + \frac{1}{\lambda_0 \mathrm{Ma}_1}\left(\bar{\Pi}_v - \frac{3}{2}\bar{\kappa}_m \bar{u}_v \frac{\mathrm{d}\bar{\rho}}{\mathrm{d}\bar{x}}\right)\left(\bar{u}_v - \frac{\gamma}{\lambda_0}\frac{\bar{\kappa}_m}{\bar{\rho}}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}\bar{x}}\right) + \frac{1}{\lambda_0 \mathrm{Ma}_1}\bar{q}_v^{(RNS)} - \frac{1}{(\gamma-1)}\bar{h}_0 = 0, \qquad (8)$$

$$\bar{m}_0 = \gamma \mathrm{Ma}_1, \quad \bar{\mathrm{p}}_0 = \frac{1}{\gamma \mathrm{Ma}_1}\left(1 + \gamma \mathrm{Ma}_1^2\right), \quad \bar{\mathrm{h}}_0 = 1 + \frac{(\gamma-1)}{2}\mathrm{Ma}_1^2.$$

Shock Density Profiles: with $\kappa_{m_0} = 1$ and $s = 0.75$



Shock Structure Parameters:



Acknowledgements and References

■ Authors would like to gratefully acknowledge the funding from Engineering and Physical Sciences Research Council (EPSRC), UK, through Grant No. EP/R008027/1.

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